Mathematical Proofs

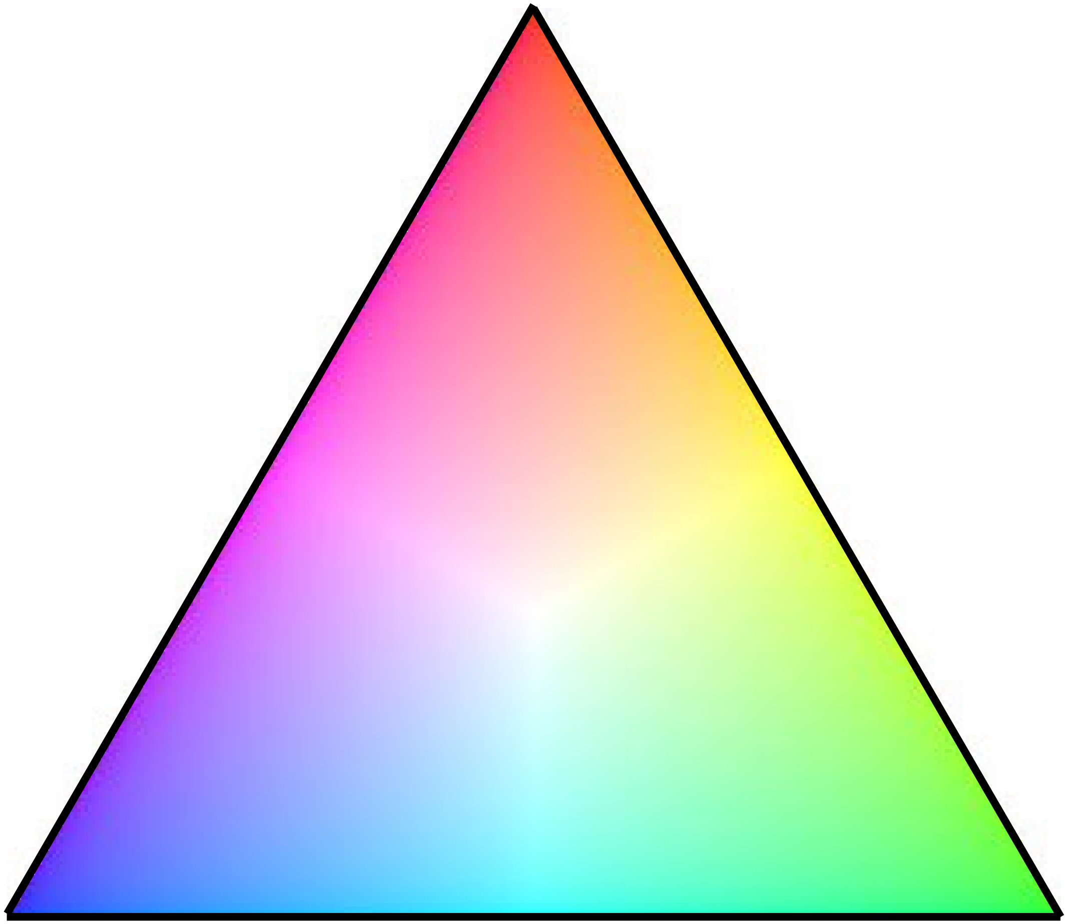
Outline for Today

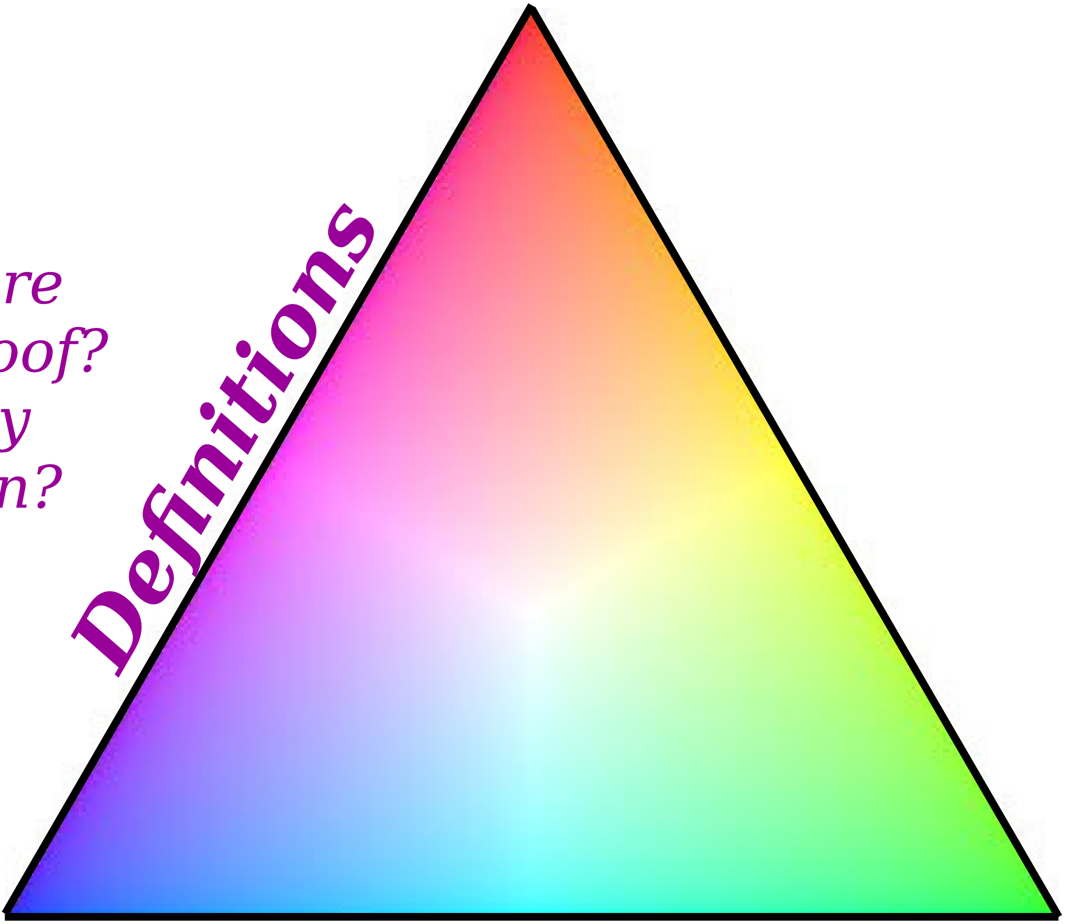
- ***How to Write a Proof***
  - Synthesizing definitions, intuitions, and conventions.
- ***Proofs on Numbers***
  - Working with odd and even numbers.
- ***Universal and Existential Statements***
  - Two important classes of statements.
- ***Proofs on Sets***
  - From Venn diagrams to rigorous math.
- ***Subsets and Set Equality***
  - Reasoning about how groups relate.

What is a Proof?

A ***proof*** is an argument that demonstrates why a conclusion is true, subject to certain standards of truth.

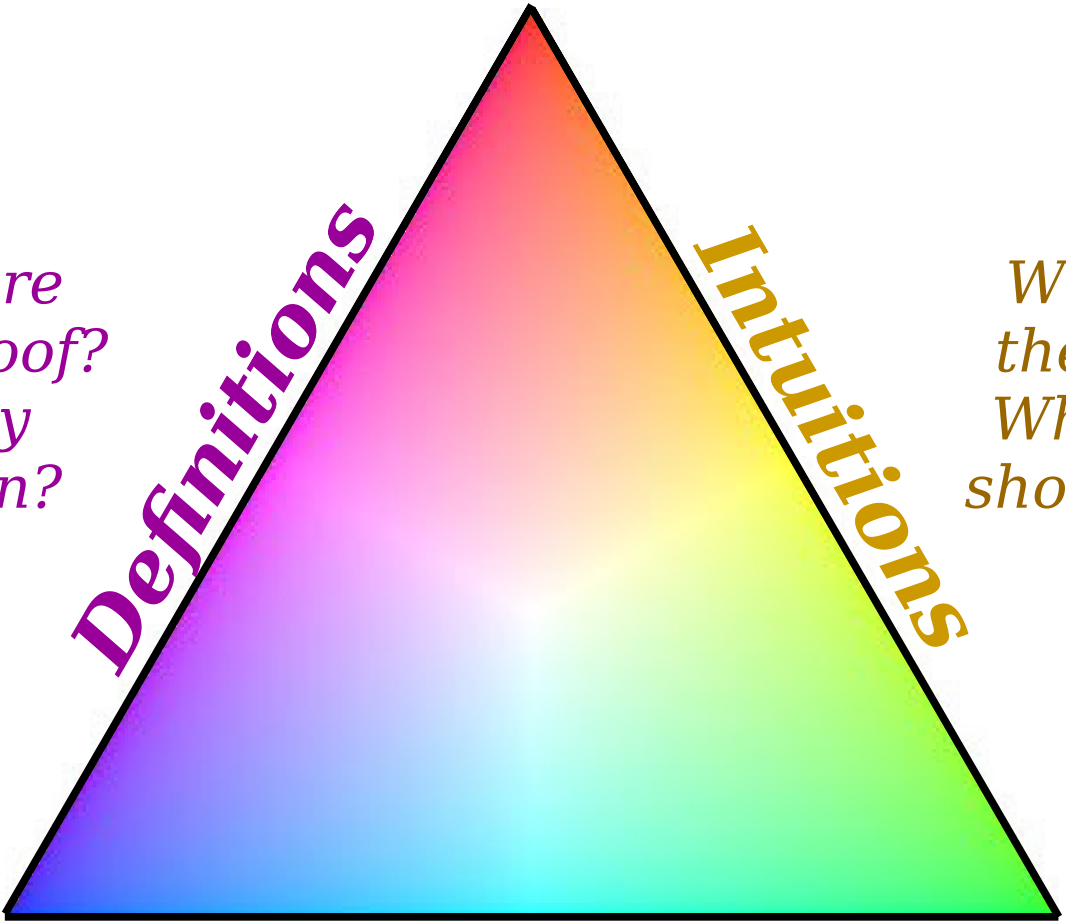
A ***mathematical proof*** is an argument that demonstrates why a mathematical statement is true, following the rules of mathematics.





Definitions

What terms are used in this proof? What do they formally mean?



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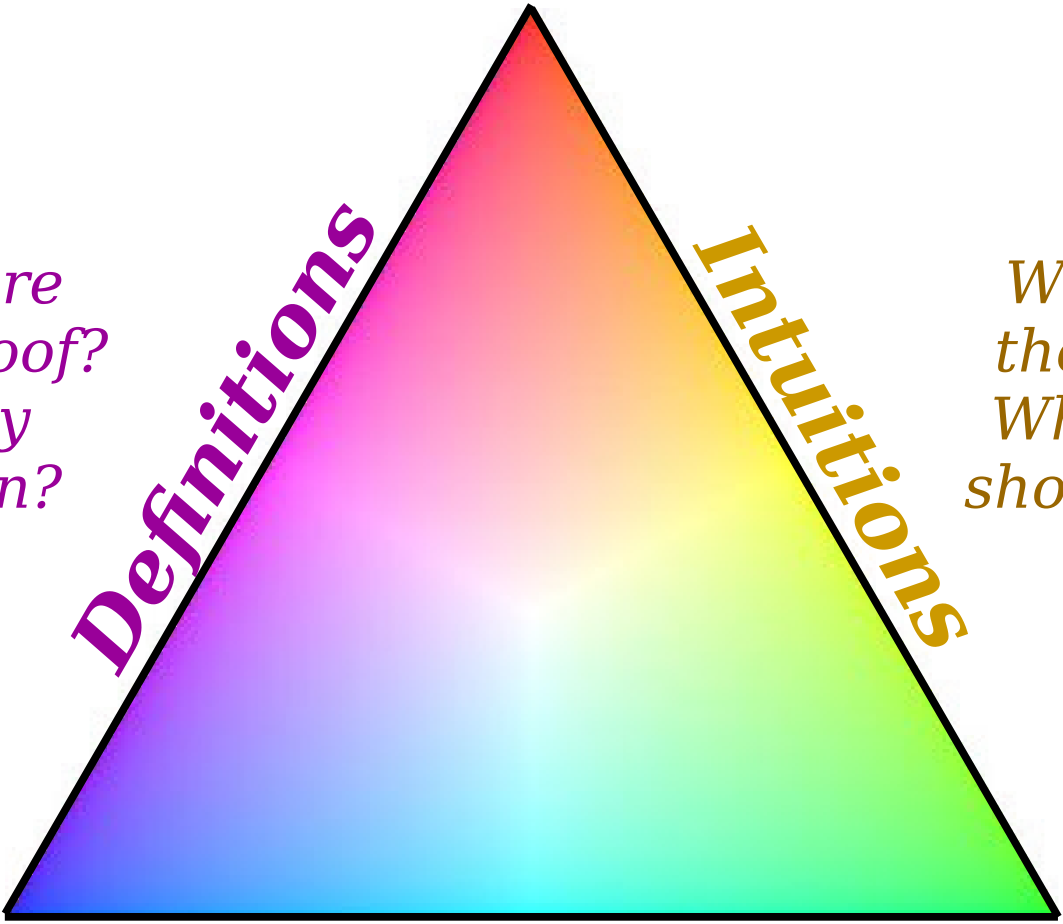

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What are the techniques
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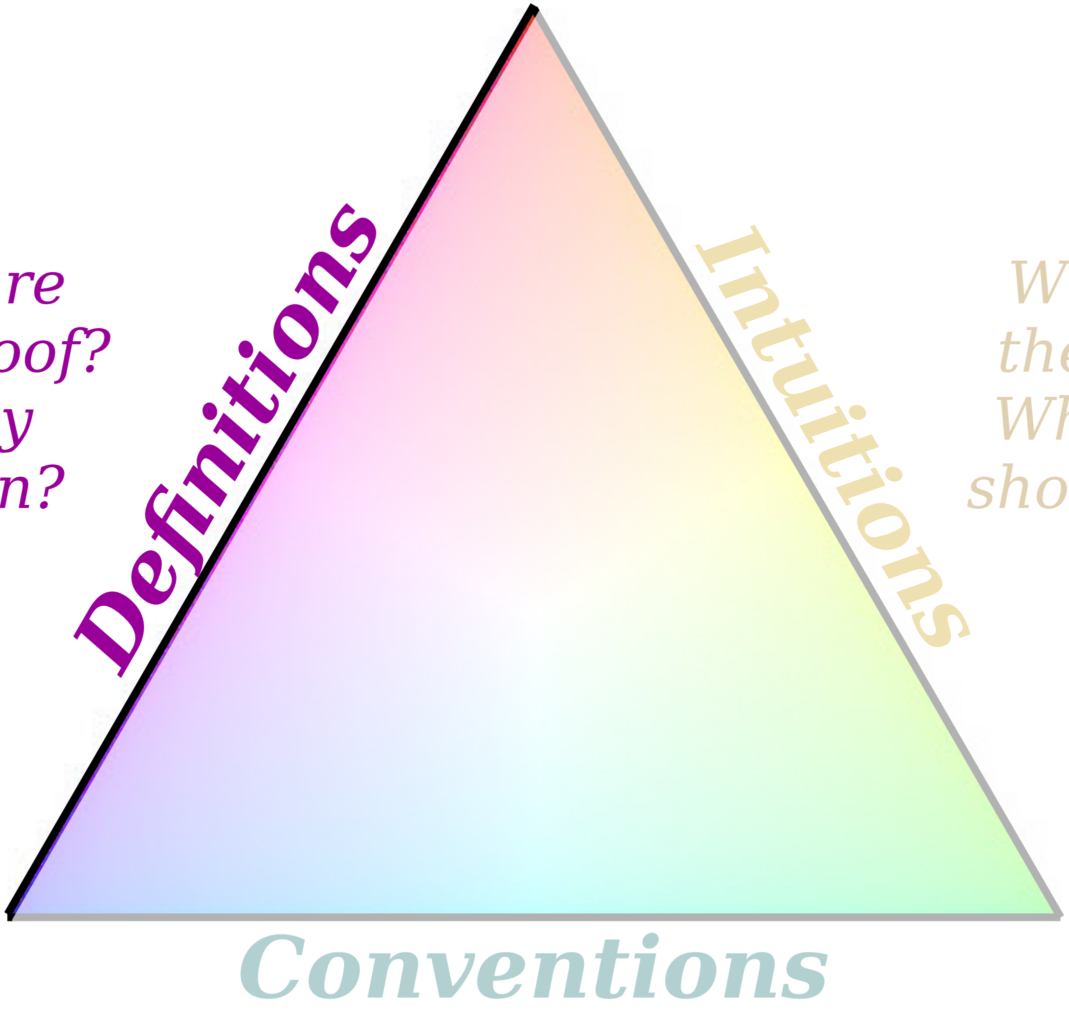
Writing our First Proof

Theorem: If n is an even integer, then n^2 is even.



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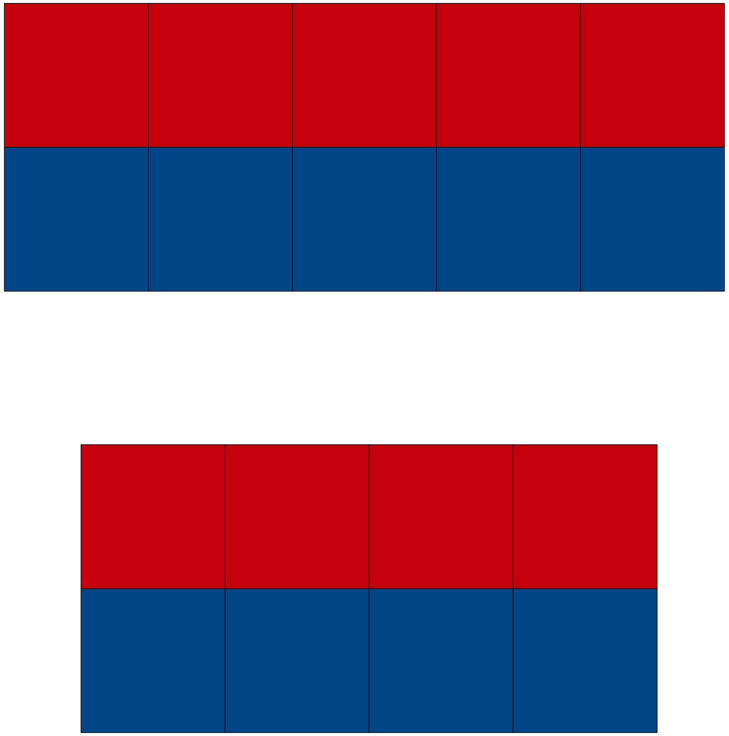
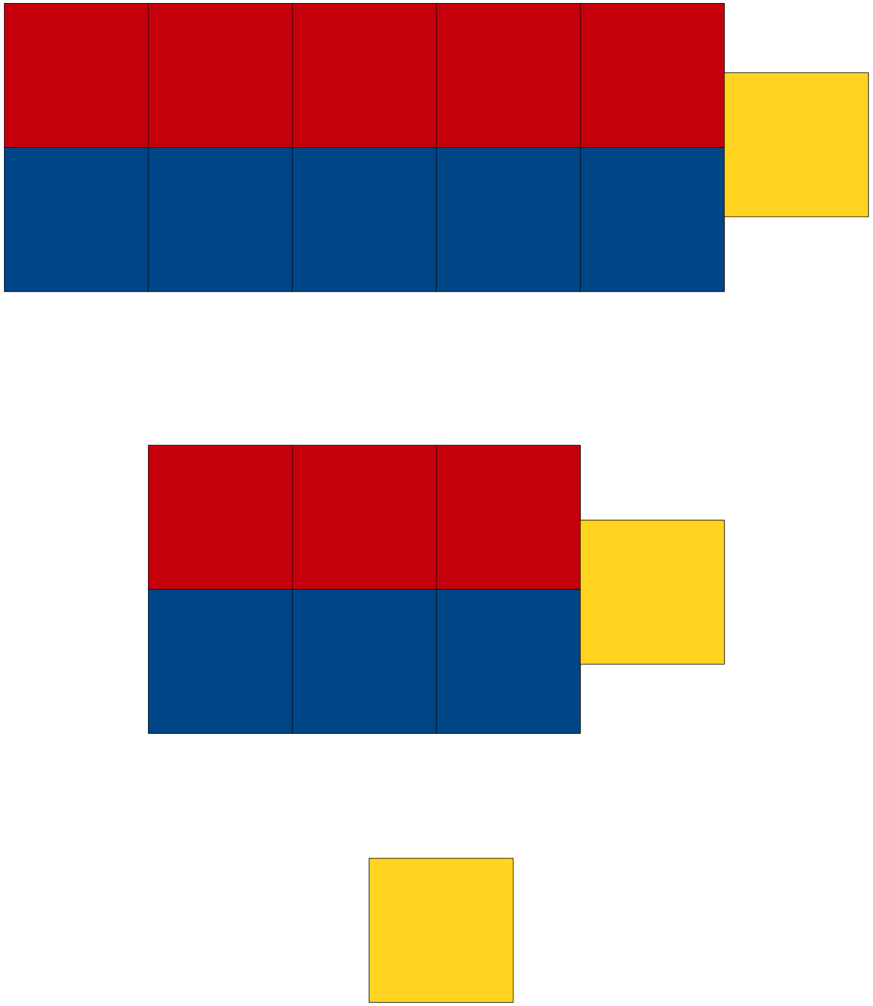


10 $2 \cdot \mathbf{5}$

8 $2 \cdot \mathbf{4}$

0 $2 \cdot \mathbf{0}$

An integer n is called ***even*** if
there is an integer k where $n = 2k$.



11 $2 \cdot \mathbf{5} + \mathbf{1}$

7 $2 \cdot \mathbf{3} + \mathbf{1}$

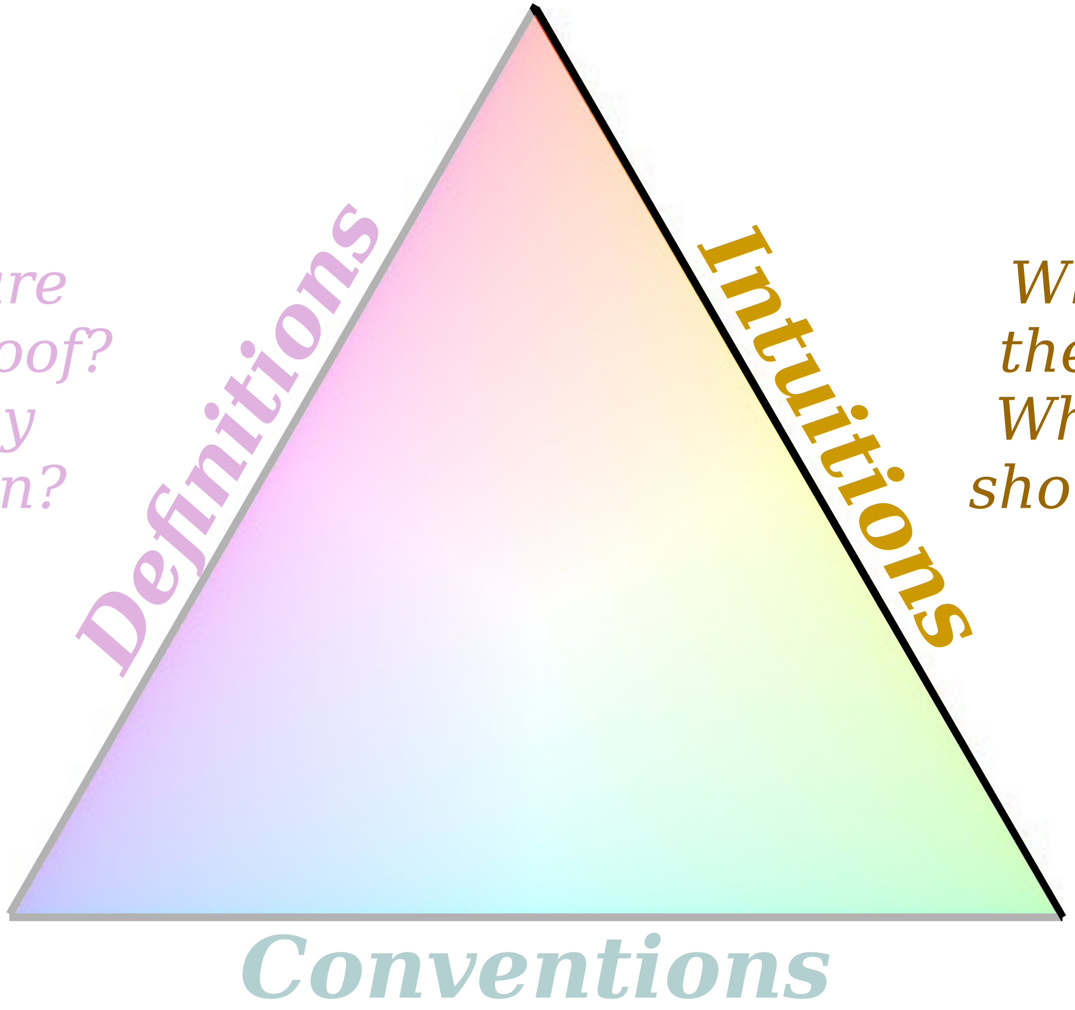
1 $2 \cdot \mathbf{0} + \mathbf{1}$

An integer n is called ***odd*** if
there is an integer k where $n = 2k+1$.

Going forward, we’ll assume the following:

1. Every integer is either even or odd.
2. No integer is both even and odd.

Theorem: If n is an even integer, then n^2 is even.



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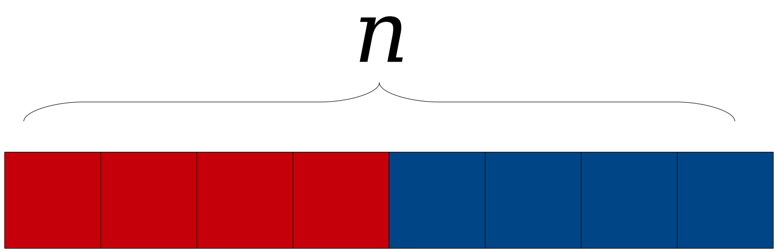
Let’s Try Some Examples!

$$\begin{aligned}
2^2 &= 4 &&= 2 \cdot \mathbf{2} \\
10^2 &= 100 &&= 2 \cdot \mathbf{50} \\
0^2 &= 0 &&= 2 \cdot \mathbf{0} \\
(\text{-}8)^2 &= 64 &&= 2 \cdot \mathbf{32} \\
n^2 & &&= 2 \cdot \mathbf{?}
\end{aligned}$$

What’s the pattern?
How do we predict this?

Theorem: If n is an even integer, then n^2 is even.

Let’s Draw Some Pictures!



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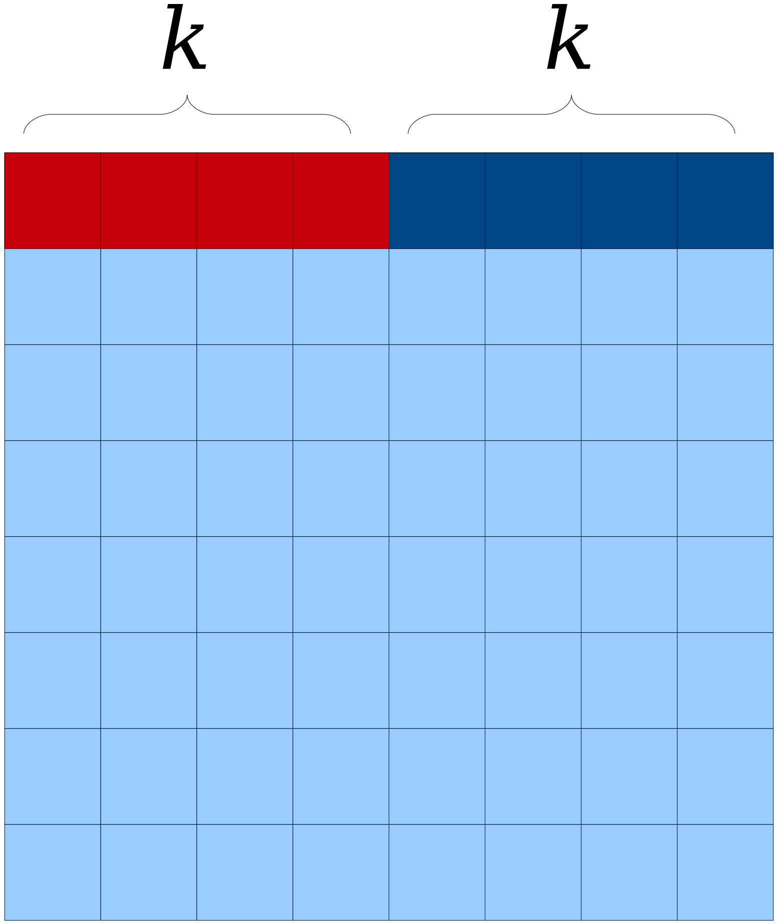

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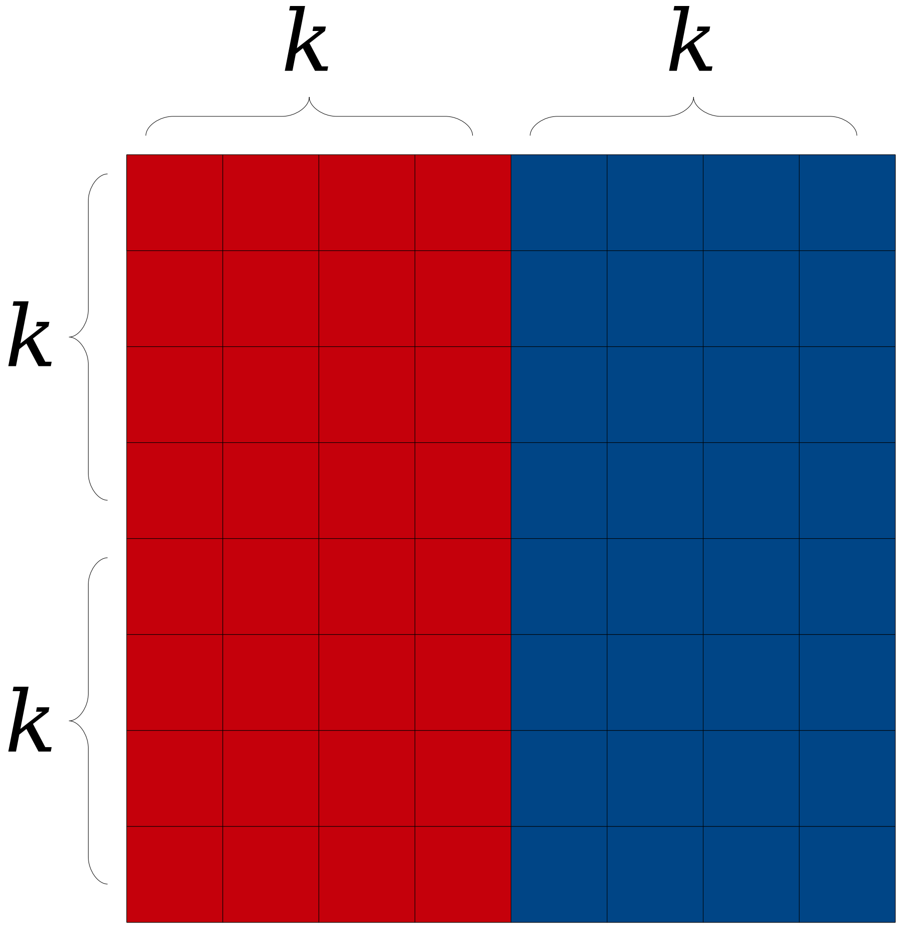

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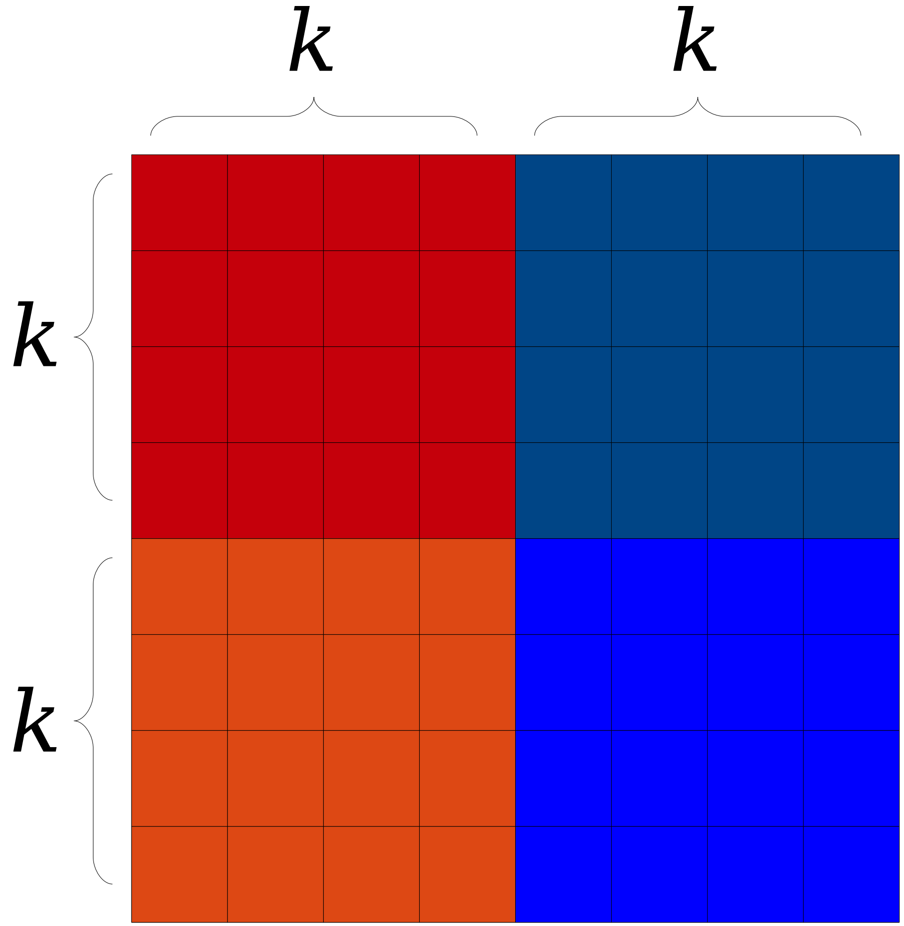
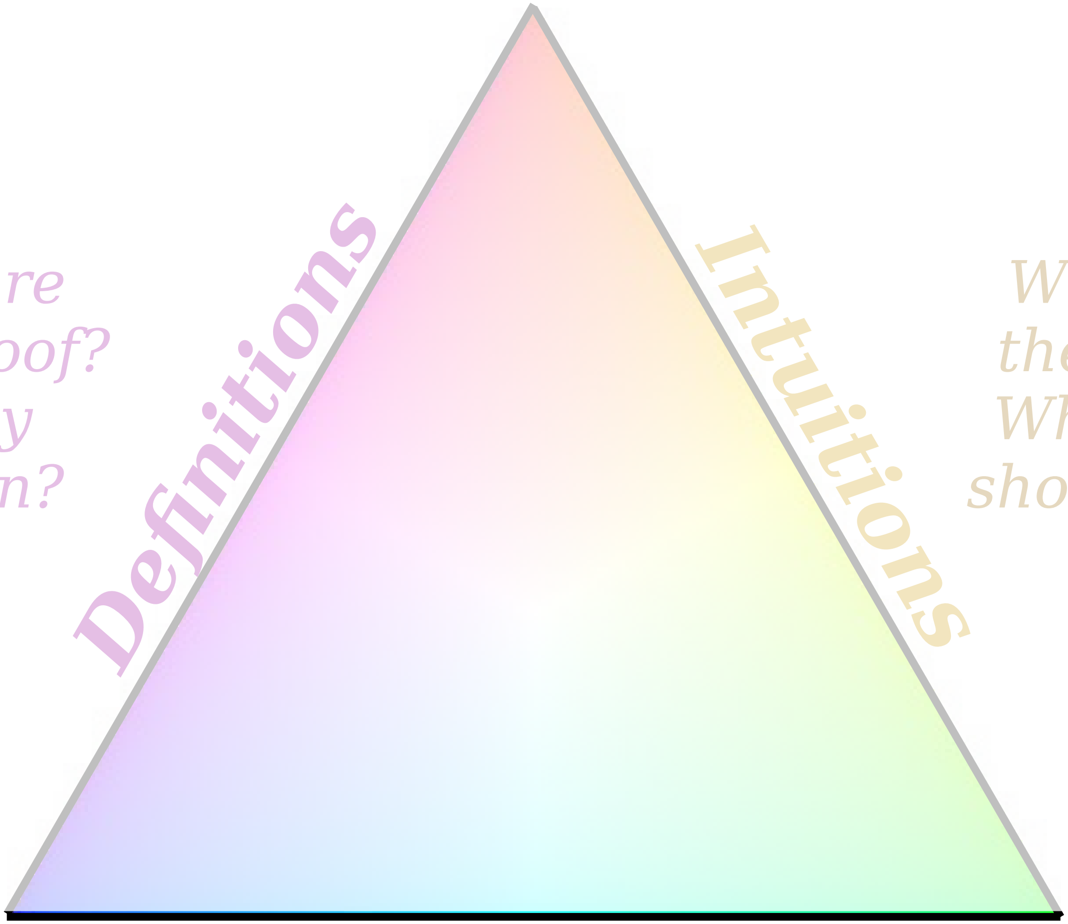


$$n^2 = 2(2k^2)$$

Theorem: If n is an even integer, then n^2 is even.



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This symbol means “end of proof”

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This is the definition of an even integer. We need to use this definition to make this proof rigorous.

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Notice how we use the value of $\boldsymbol{k}$ that we obtained above. Giving names to quantities, even if we aren't fully sure what they are, allows us to manipulate them. This is similar to variables in programs.

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Theorem: If *n* … en.
Proof: Let *n* b…

Since *n* …
such th…

This me… ^{2}).

Our ultimate goal is to prove that **n^2** is even. This means that we need to find some **m** such that **$n^2 = 2m$**. Here, we're explicitly showing how we can do that.

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Hey, that's what we were trying to show!
We're done now.

Therefore, n^2 is even. ■

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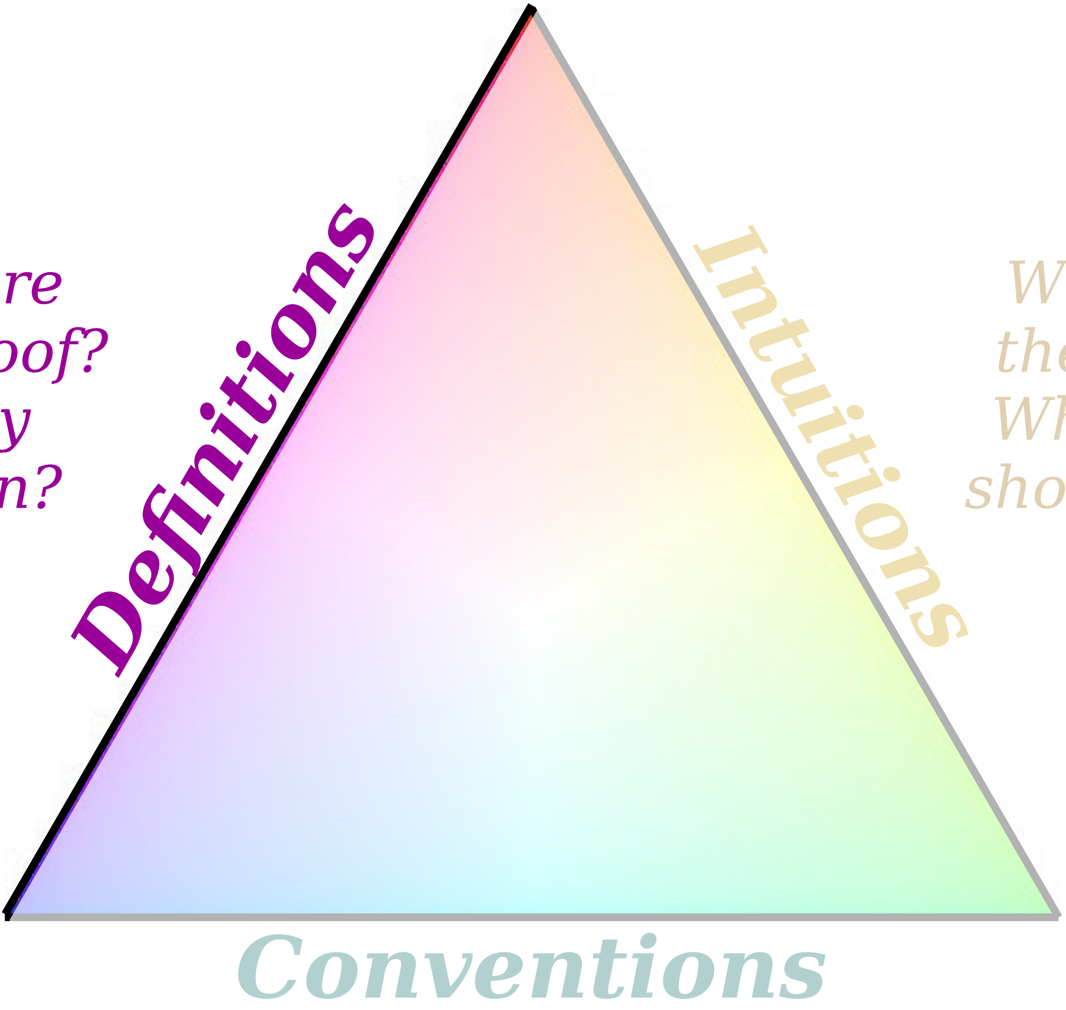
Our Next Proof

Theorem: For any integers m and n,
if m and n are odd, then $m + n$ is even.



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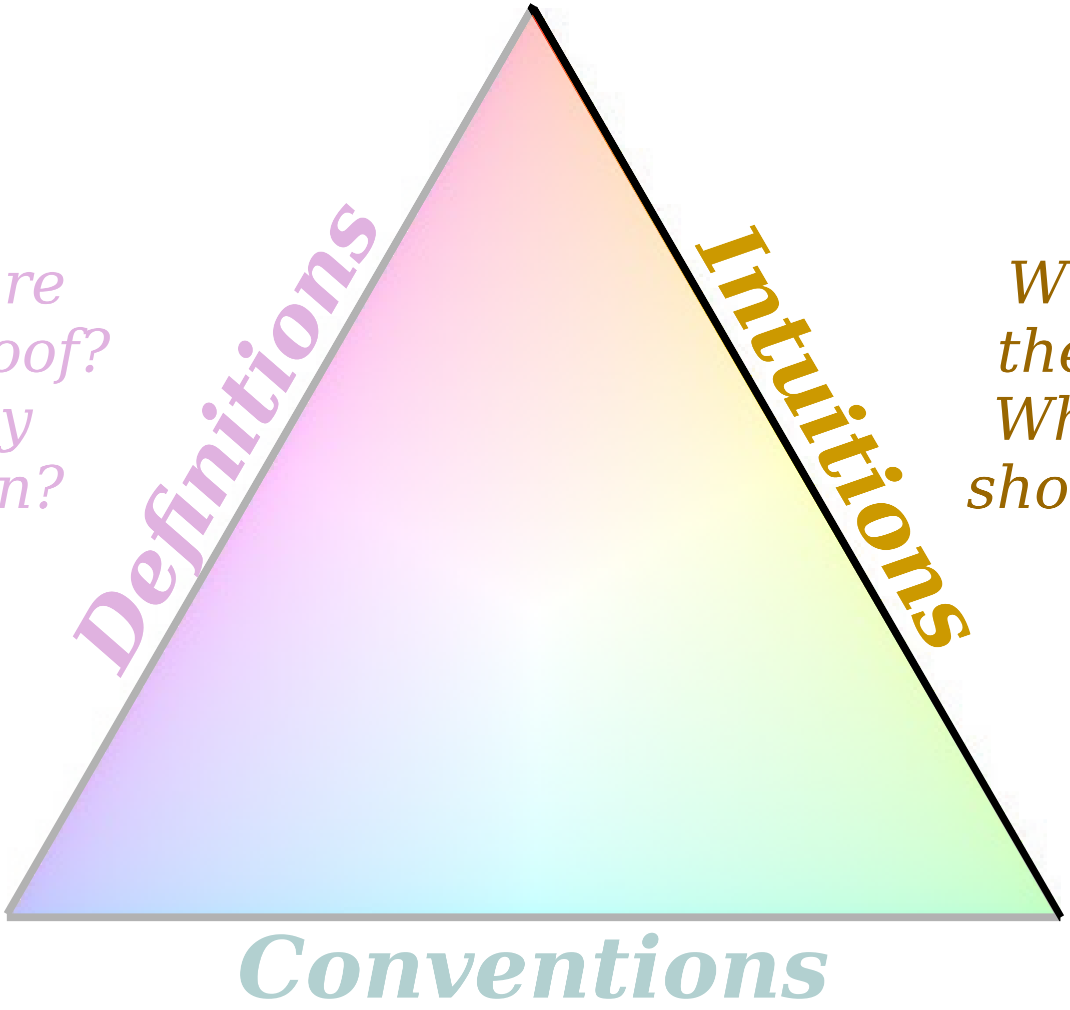

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Let’s Try Some Examples!

$$\begin{array}{ccccc} 1 + 1 & = & 2 & = & 2 \cdot \mathbf{1} \\ 137 + 103 & = & 240 & = & 2 \cdot \mathbf{120} \\ -5 + 5 & = & 0 & = & 2 \cdot \mathbf{0} \\ m + n & & & = & 2 \cdot \mathbf{?} \end{array}$$

What’s the pattern? How do we predict this?

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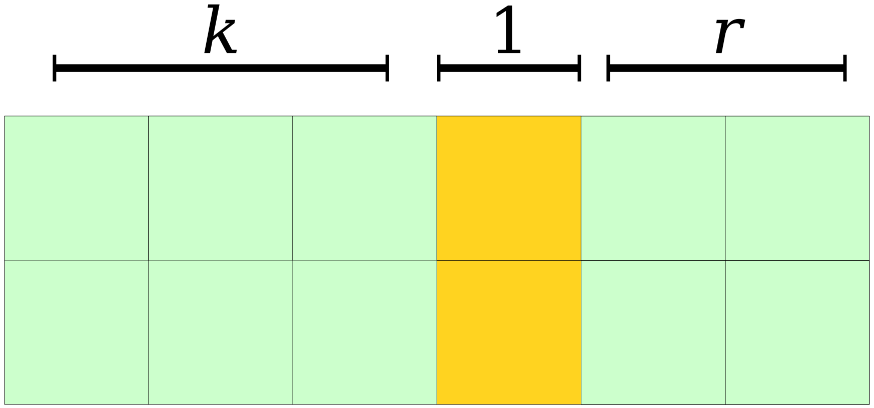

$2k+1$

$2r+1$

$$(2k+1) + (2r+1) = 2(k + r + 1)$$

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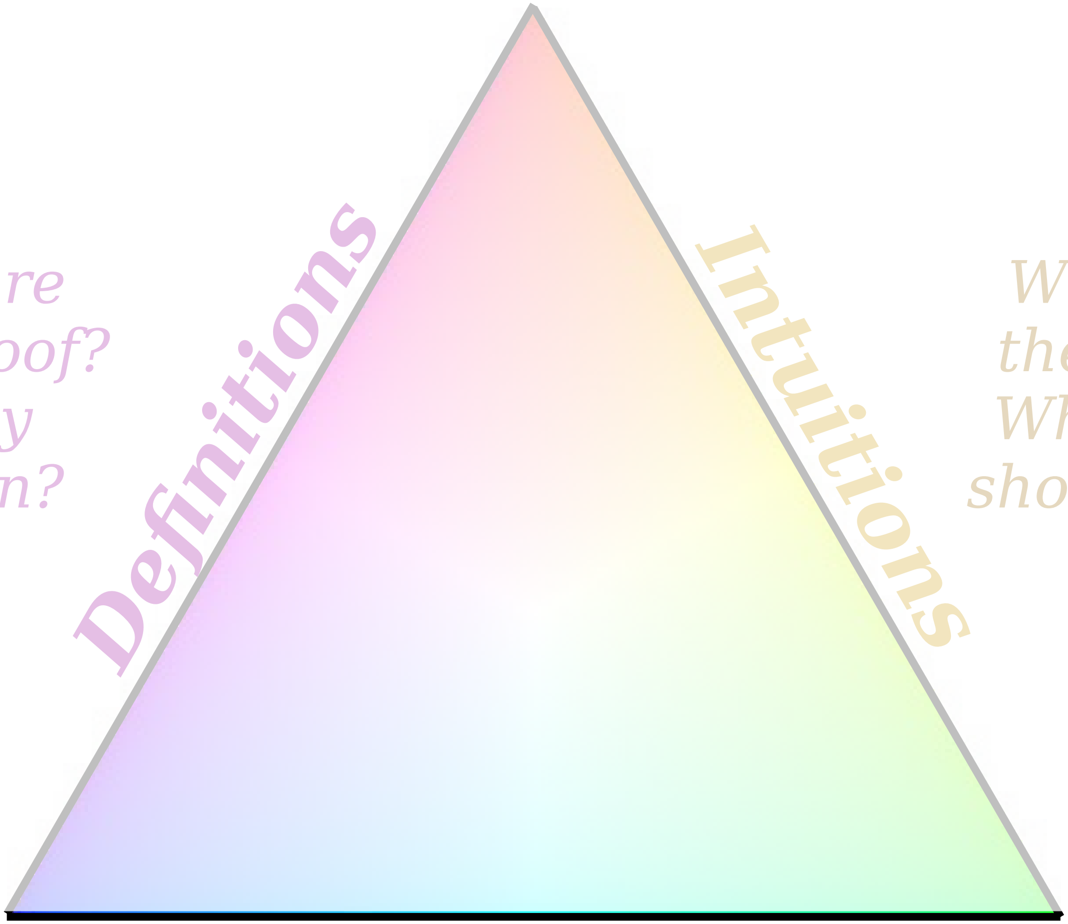

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Proof: Consider any arbitrary integers m and n where m and n are odd. Since m is odd, we know that there is an integer k where

$$m = 2k + 1. \tag{1}$$

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Similarly, because n is odd there must be some integer r such that

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This is called making ***arbitrary choices***. Rather than specifying what ***m*** and ***n*** are, we're signaling to the reader that they could, in principle, supply any choices of ***m*** and ***n*** that they'd like.

By picking ***m*** and ***n*** arbitrarily, anything we prove about ***m*** and ***n*** will generalize to all possible choices we could have made.

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Numbering these equalities lets us refer back to them later on, making the flow of the proof a bit easier to understand.

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This is a complete sentence! Proofs are expected to be written in complete sentences, so you'll often use punctuation at the end of formulas.

We recommend using the "mugga mugga" test – if you read a proof and replace all the mathematical notation with "mugga mugga," what comes back should be a valid sentence.

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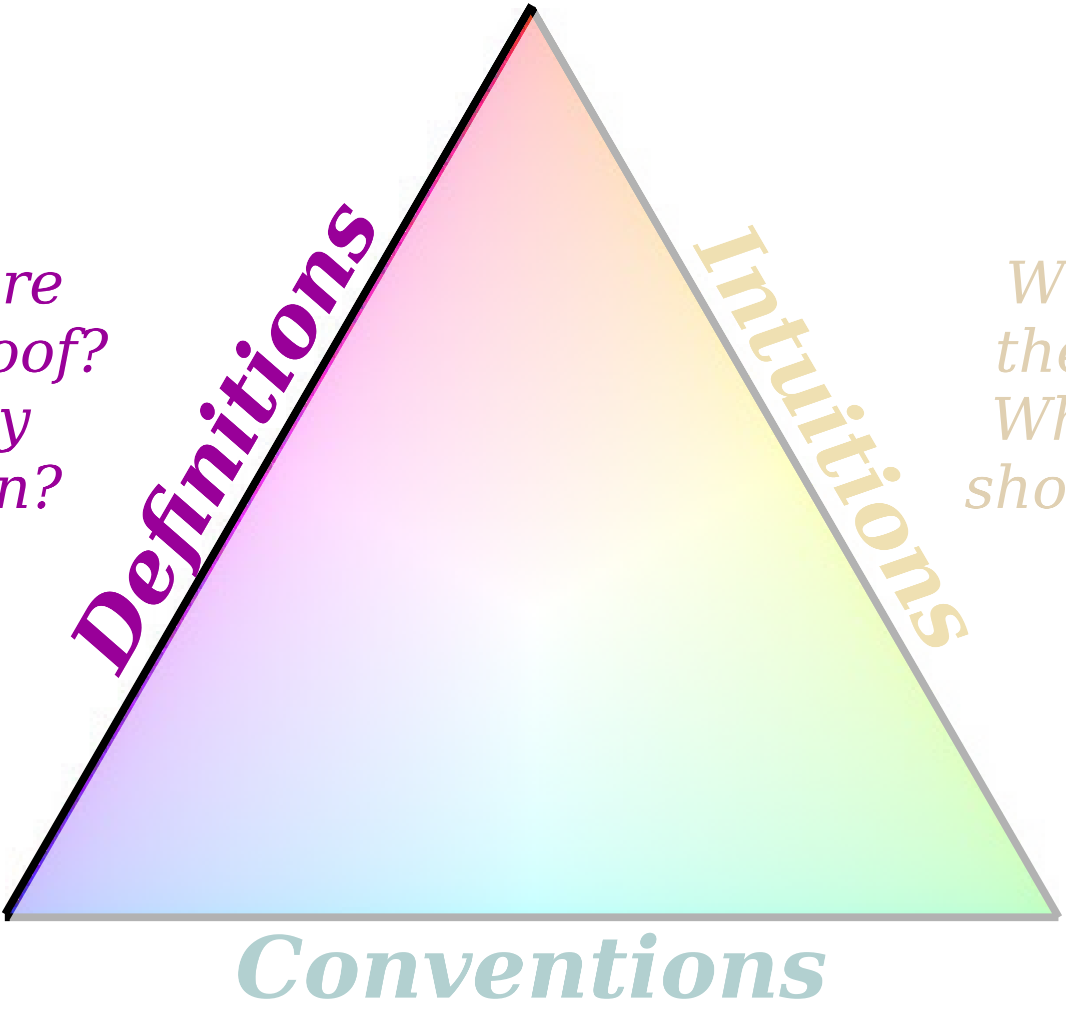
Some Little Exercises

- Here's a list of other theorems that are true about odd and even numbers:
  - ***Theorem:*** The sum and difference of any two even numbers is even.
  - ***Theorem:*** The sum and difference of an odd number and an even number is odd.
  - ***Theorem:*** The product of any integer and an even number is even.
  - ***Theorem:*** The product of any two odd numbers is odd.
- Going forward, we'll just take these results for granted. Feel free to use them in the problem sets.
- If you'd like to practice the techniques from today, try your hand at proving these results!

Universal and Existential Statements

Theorem: For any odd integer n,
there exist integers r and s where $r^2 - s^2 = n$.



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This result is true for every possible choice of odd integer *n*. It’ll work for *n* = 1, *n* = 137, *n* = 103, etc.

Theorem: For any odd integer n,
there exist integers r and s where $r^2 - s^2 = n$.

We aren't saying this is true for every choice of r and s. Rather, we're saying that ***somewhere out there*** are choices of r and s where this works.

Universal vs. Existential Statements

- A ***universal statement*** is a statement of the form

For all *x*, `[some-property]` holds for *x*.

- We've seen how to prove these statements.
- An ***existential statement*** is a statement of the form

There is some *x* where `[some-property]` holds for *x*.

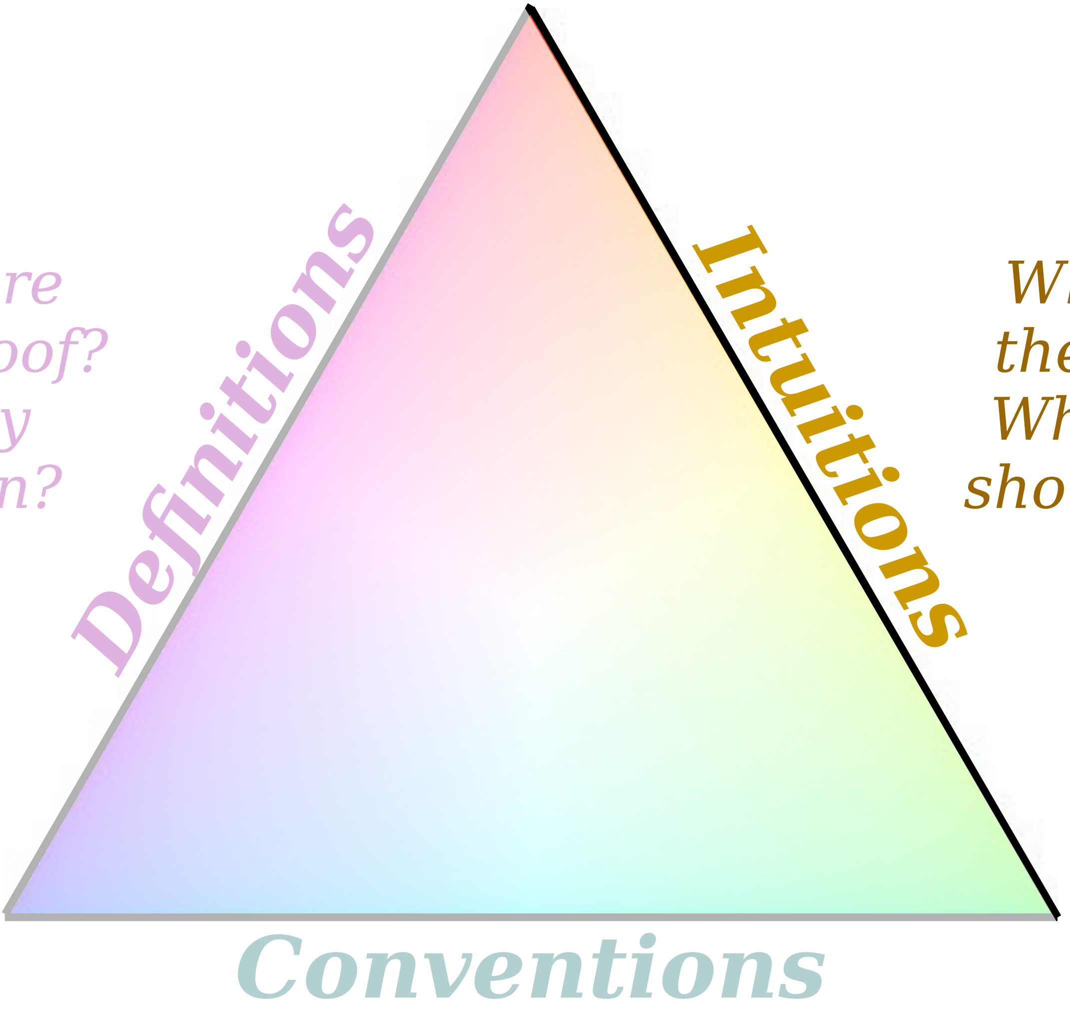
- How do you prove an existential statement?

Proving an Existential Statement

- Over the course of the quarter, we will see several different ways to prove an existential statement of the form

There is an *x* where `[some-property]` holds for *x*.

- ***Simplest approach:*** Search far and wide, find an *x* that has the right property, then show why your choice is correct.



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Let’s Try Some Examples!

$1 = \underline{\qquad}^2 - \underline{\qquad}^2$

$3 = \underline{\qquad}^2 - \underline{\qquad}^2$

$5 = \underline{\qquad}^2 - \underline{\qquad}^2$

$7 = \underline{\qquad}^2 - \underline{\qquad}^2$

$9 = \underline{\qquad}^2 - \underline{\qquad}^2$

Question: Fill in these blanks and see if you can come up with a pattern for why this result is true.

Respond at pollev.com/zhenglian740

Theorem: For any odd integer n, there exist integers r and s where $r^2 - s^2 = n$.

Let’s Try Some Examples!

$$1 = \mathbf{1}^2 - \mathbf{0}^2$$

$$3 = \mathbf{2}^2 - \mathbf{1}^2$$

$$5 = \mathbf{3}^2 - \mathbf{2}^2$$

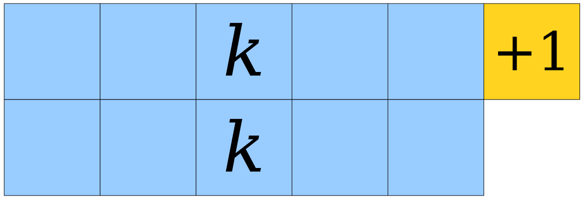
$$7 = \mathbf{4}^2 - \mathbf{3}^2$$

$$9 = \mathbf{5}^2 - \mathbf{4}^2$$

We’ve got a pattern – but why does this work?

Theorem: For any odd integer n, there exist integers r and s where $r^2 - s^2 = n$.

Let’s Draw Some Pictures!



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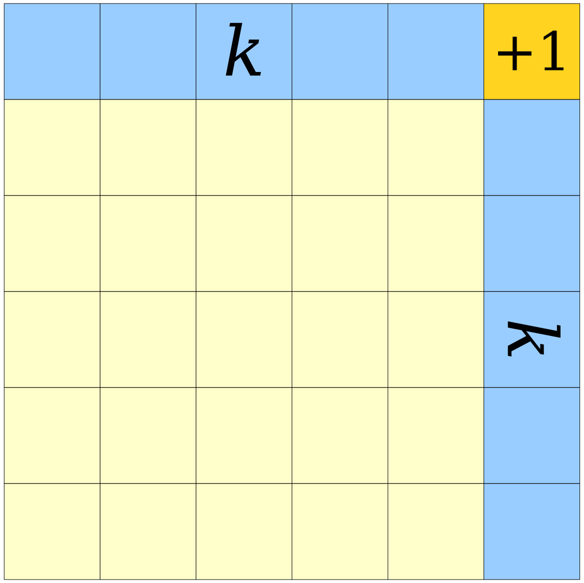
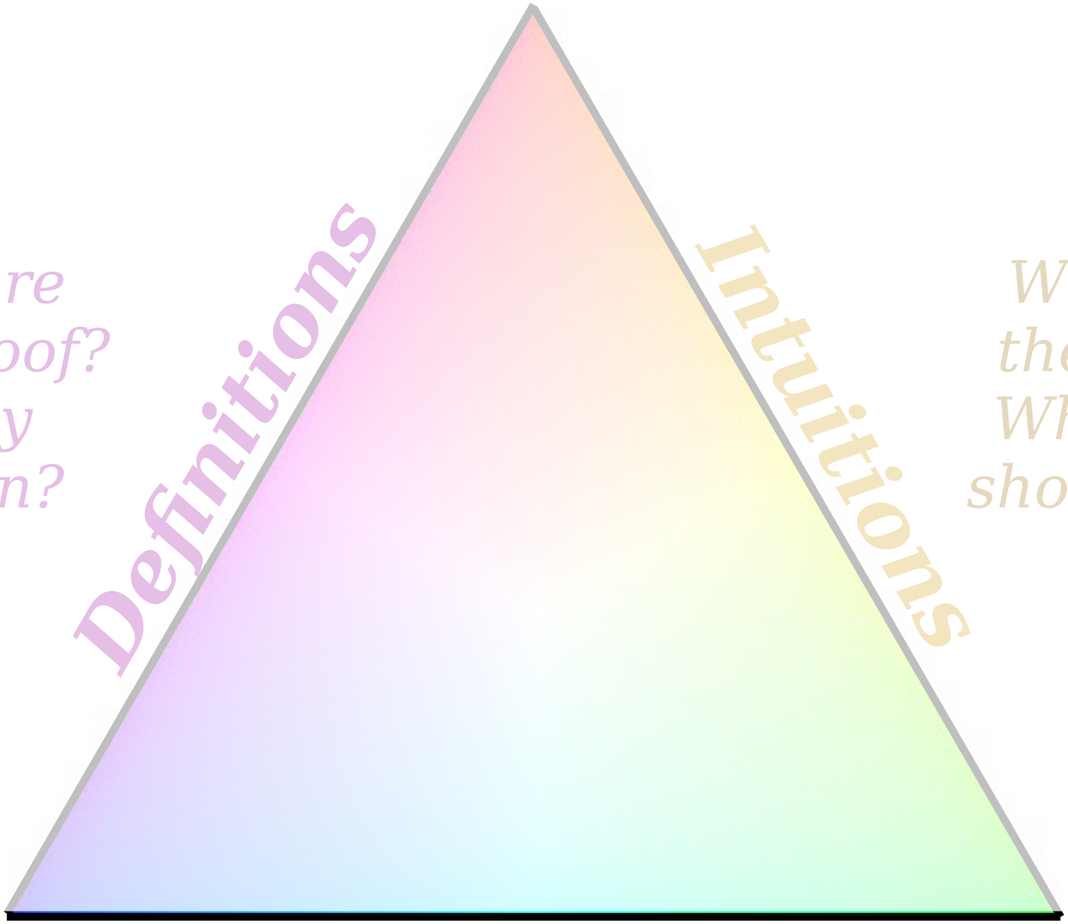


$$(k+1)^2 - k^2 = 2k+1$$

Theorem: For any odd integer n, there exist integers r and s where $r^2 - s^2 = n$.



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Now, let $r = k+1$ and $s = k$.

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Proof: Pick any odd integer n. Since n is odd, we know there is some integer k where $n = 2k + 1$.

Now, let $r = k+1$ and $s = k$. Then we see that

$$r^2 – s^2 \quad = \quad (k+1)^2 – k^2$$

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$$\begin{aligned} r^2 - s^2 &= (k+1)^2 - k^2 \\ &= k^2 + 2k + 1 - k^2 \end{aligned}$$

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Proof: Pick any odd integer n. Since n is odd, we know there is some integer k where $n = 2k + 1$.

Now, l

We make an ***arbitrary choice***. Rather than specifying what ***n*** is, we're signaling to the reader that they could, in principle, supply any choice ***n*** that they'd like.

$$= \quad 2k + 1$$

$$= \quad n.$$

This means that $r^2 - s^2 = n$, which is what we needed to show. ■

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Now, let $r = k+1$ and $s = k$. Then we see that

$$r^2 - s^2 = (k+1)^2 - k^2$$

We’re trying to prove an existential statement. The easiest way to do that is to just give concrete choices of the objects being sought out.

This mean e
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This means that $r^2 - s^2 = n$, which is what we needed to show. ■

Let’s take a quick break!

Time-Out for Announcements!

Reading Recommendations

- We’ve released two handouts online that you should read over:
  - How to Succeed in CS103
  - Guide to Proofs
- Additionally, if you haven’t yet read over the Guide to Elements and Subsets, we’d recommend doing so.

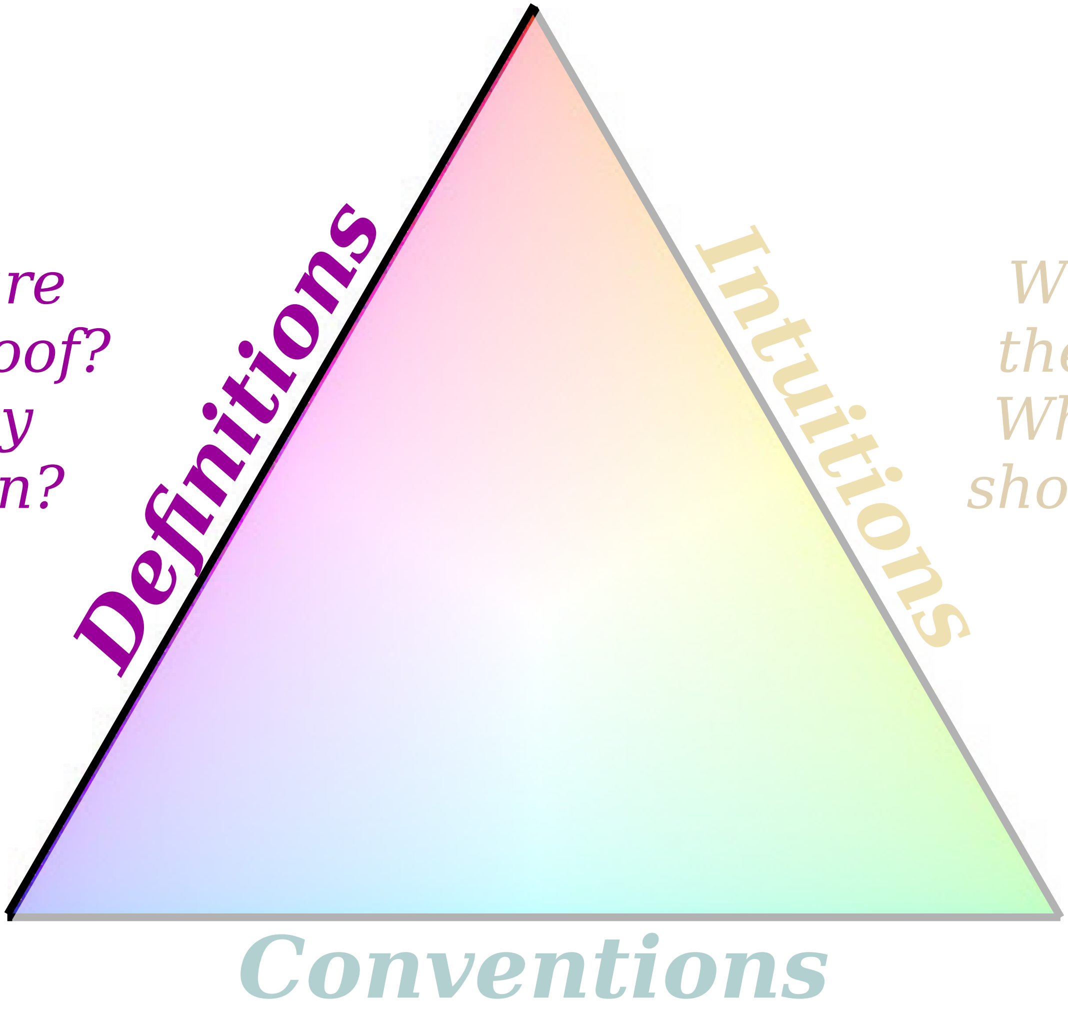
Problem Set 0

- Problem Set 0 went out on Monday. It’s due this Friday at 5:30PM.
    - Even though this just involves setting up your compiler and submitting things, please start this one early. If you start things on Friday morning, we can’t help you troubleshoot Qt Creator issues!
    - There’s a very detailed troubleshooting guide up on the CS103 website detailing common fixes. If you’re still having trouble, please feel free to ask on EdStem!
    - In-person Qt Creator help session **this Thursday**, 7-9 PM, in Durand 353

Back to CS103!

Proofs on Sets

Theorem: If A, B, and C are sets, then for any $x \in (A \cap B) \cup C$, we have $x \in (A \cup C) \cap (B \cup C)$.



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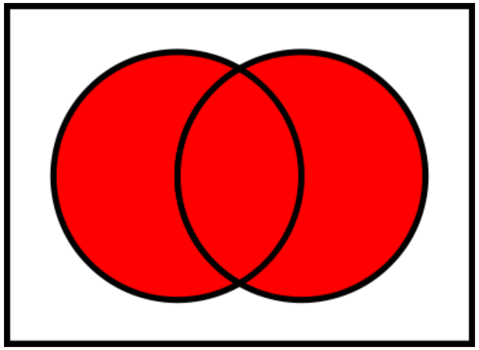
This is the ***element-of*** relation $\in$. It means that this object x is one of the items inside these sets.

Theorem: If A, B, and C are sets, then for any $x \in (A \cap B) \cup C$, we have $x \in (A \cup C) \cap (B \cup C)$.

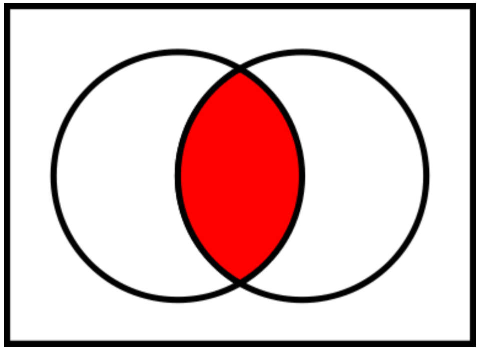
What are these, again?

Set Combinations

- In our last lecture, we saw four ways of combining sets together.



$S \cup T$



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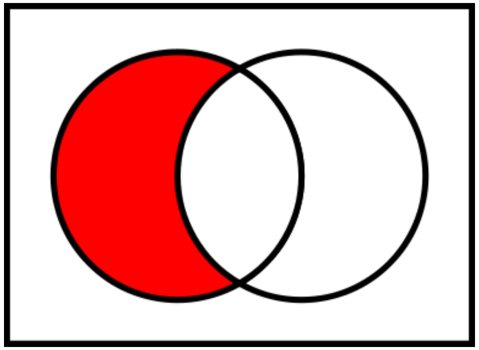
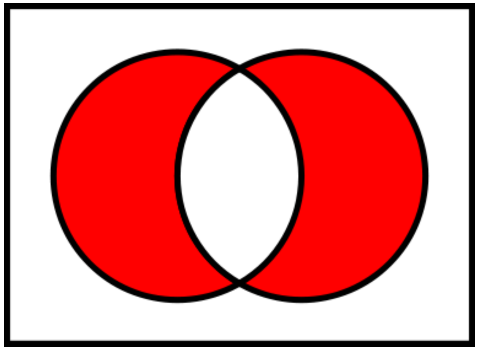


$S - T$



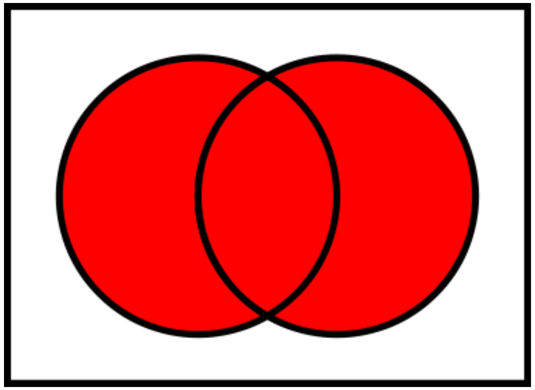
$S \Delta T$

- The above pictures give a holistic sense of how these operations work.
- However, mathematical proofs tend to work on sets in a different way.

Important Fact:

Proofs about sets *almost always* focus on individual elements of those sets. It’s rare to talk about how collections relate to one another “in general.”

Set Union



S* ∪ *T

Definition: The set $S \cup T$ is the set where, for any x:

$x \in S \cup T$ when $x \in S$ or $x \in T$ (or both)

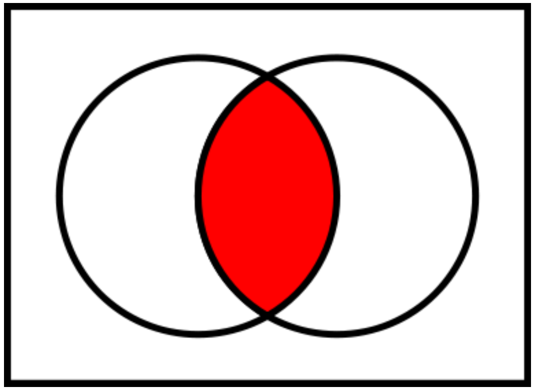
To prove that x ∈ S ∪ T:

Prove either that $x \in S$ or that $x \in T$ (or both).

If you know that x ∈ S ∪ T:

You can conclude that $x \in S$ or that $x \in T$ (or both).

Set Intersection



$S \cap T$

Definition: The set $S \cap T$ is the set where, for any x:

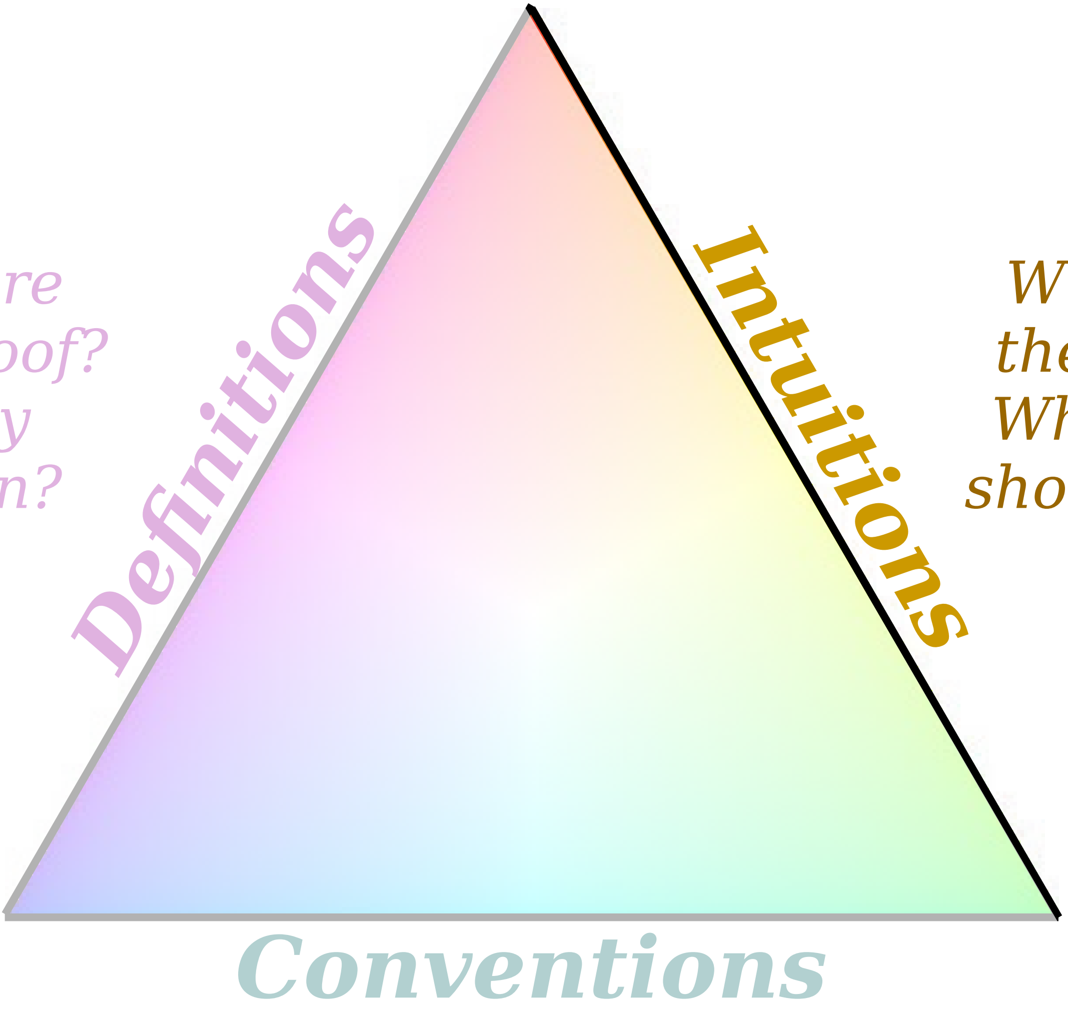
$$x \in S \cap T \qquad \text{when} \qquad x \in S \text{ and } x \in T$$

To prove that $x \in S \cap T$:

Prove both that $x \in S$ and that $x \in T$.

If you know that $x \in S \cap T$:

You can conclude both that $x \in S$ and that $x \in T$.



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Let’s Try Some Examples!

$A = \{1, 2, 3\}$
$B = \{2, 3, 4\}$
$C = \{3, 4, 5\}$

Theorem: If A, B, and C are sets, then for any $x \in (A \cap B) \cup C$, we have $x \in (A \cup C) \cap (B \cup C)$.

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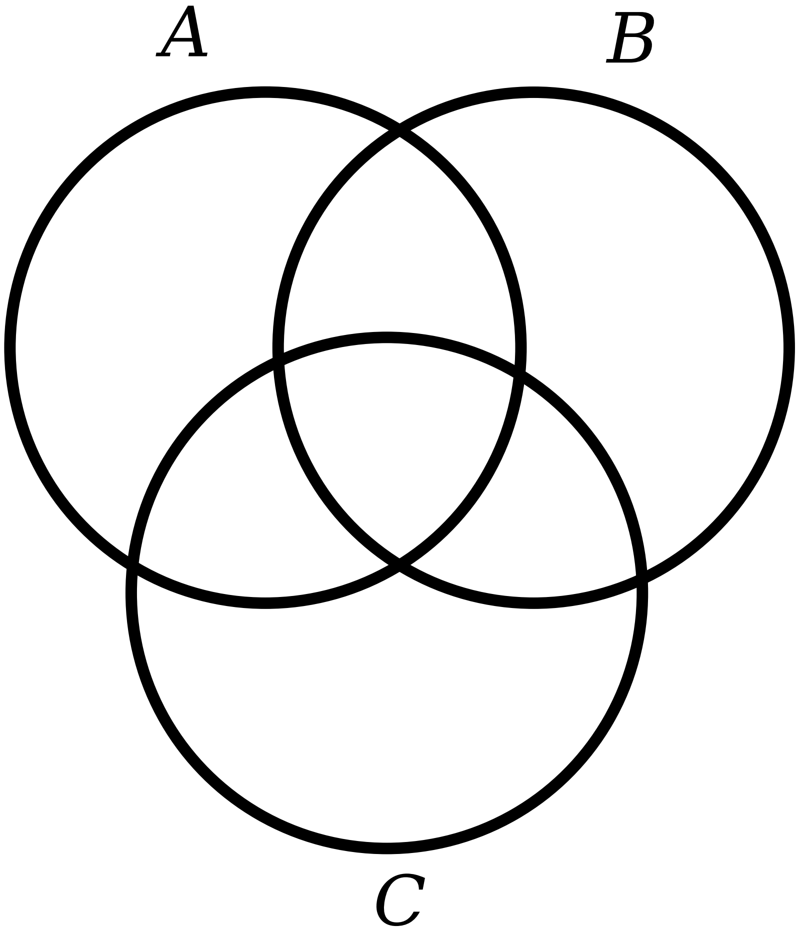

Question: Pick $x = 1$.
Is $x \in (A \cap B) \cup C$?
Is $x \in (A \cup C) \cap (B \cup C)$?

Now pick $x = 2$.
Is $x \in (A \cap B) \cup C$?
Is $x \in (A \cup C) \cap (B \cup C)$?

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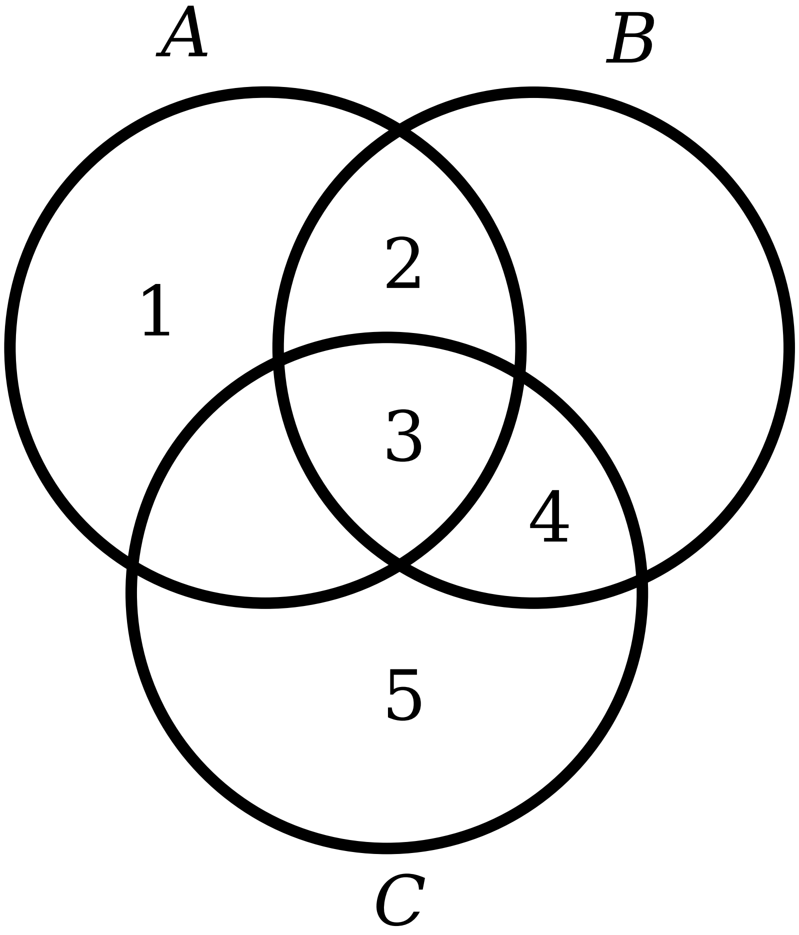

$x = 1$

✓ ✗ ✗
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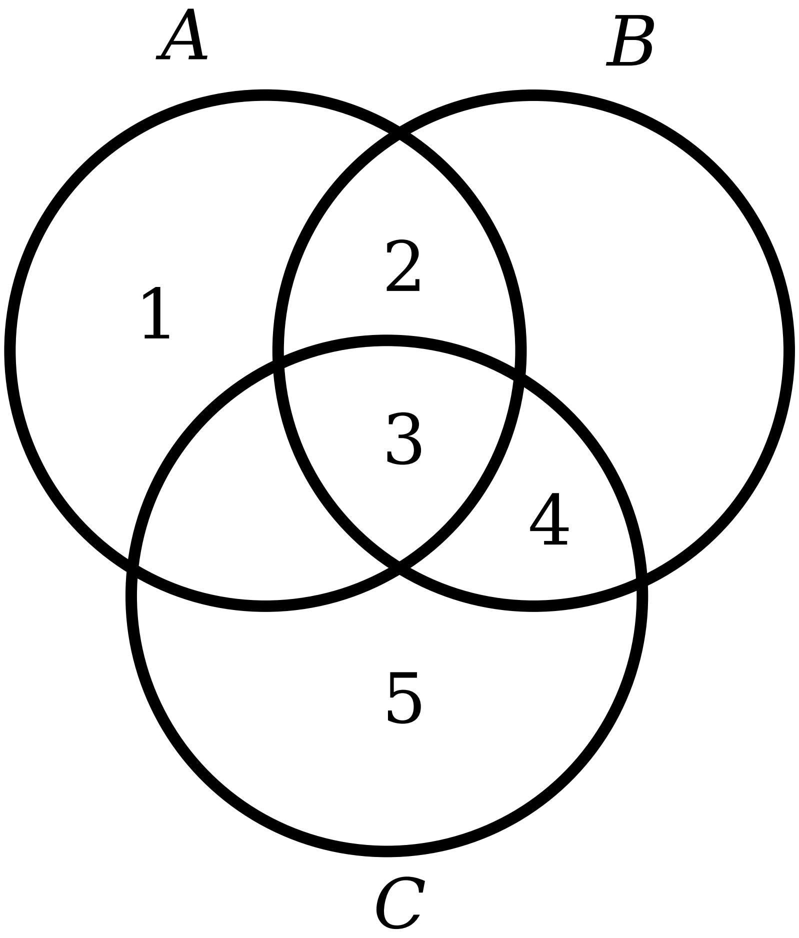

$x = 2$

Is $x \in (\overset{\checkmark}{A} \cap \overset{\checkmark}{B}) \cup \overset{\times}{C}$?

Is $x \in (\overset{\checkmark}{A} \cup \overset{\times}{C}) \cap (\overset{\checkmark}{B} \cup \overset{\times}{C})$?

Theorem: If A, B, and C are sets, then for any $x \in (A \cap B) \cup C$, we have $x \in (A \cup C) \cap (B \cup C)$.

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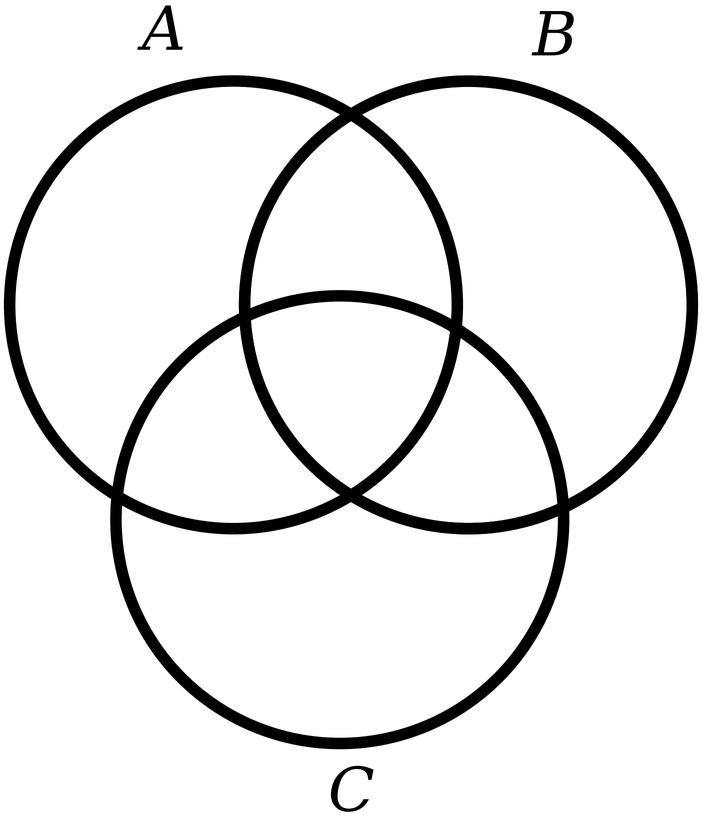

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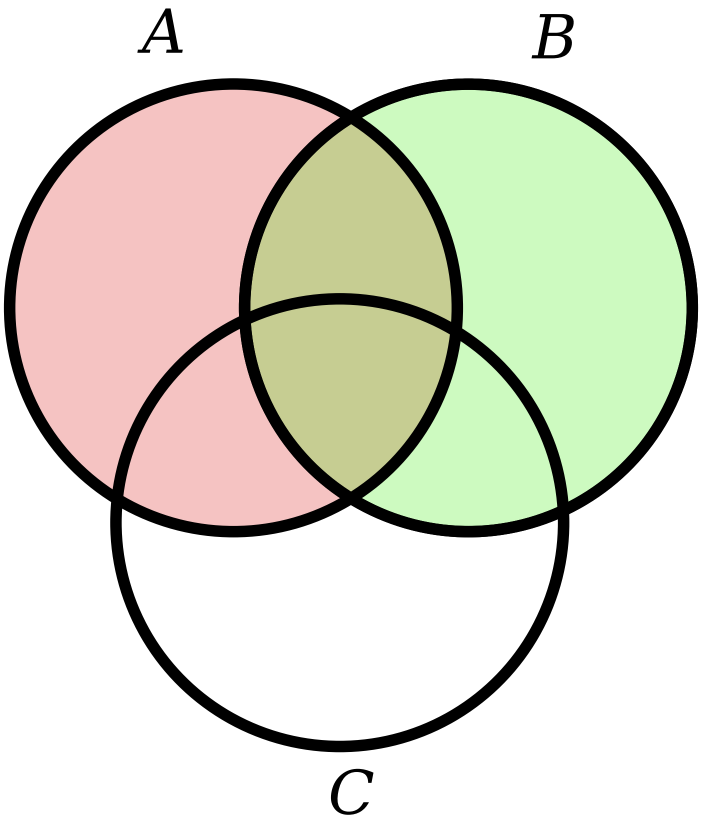

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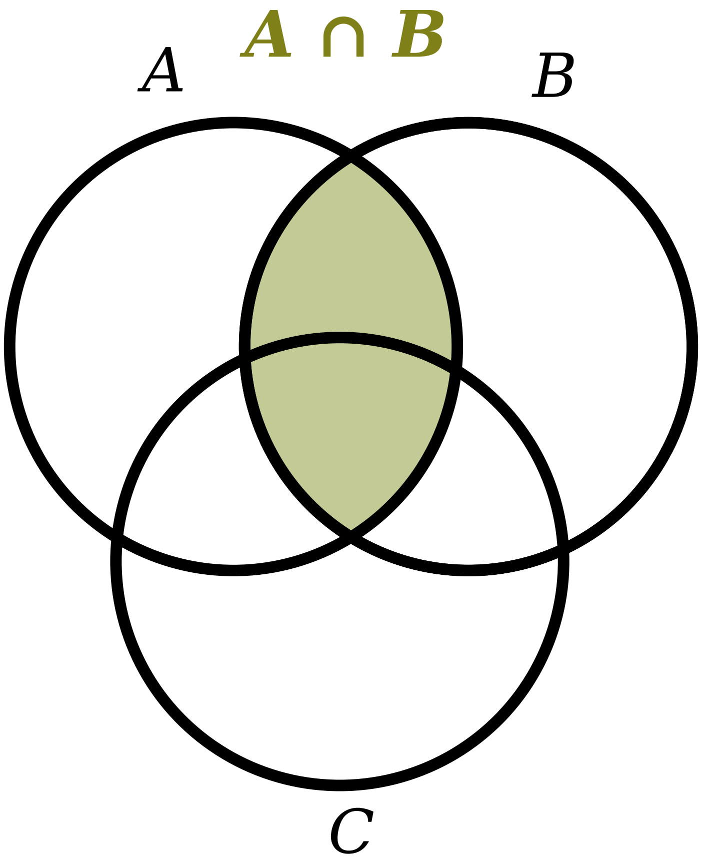

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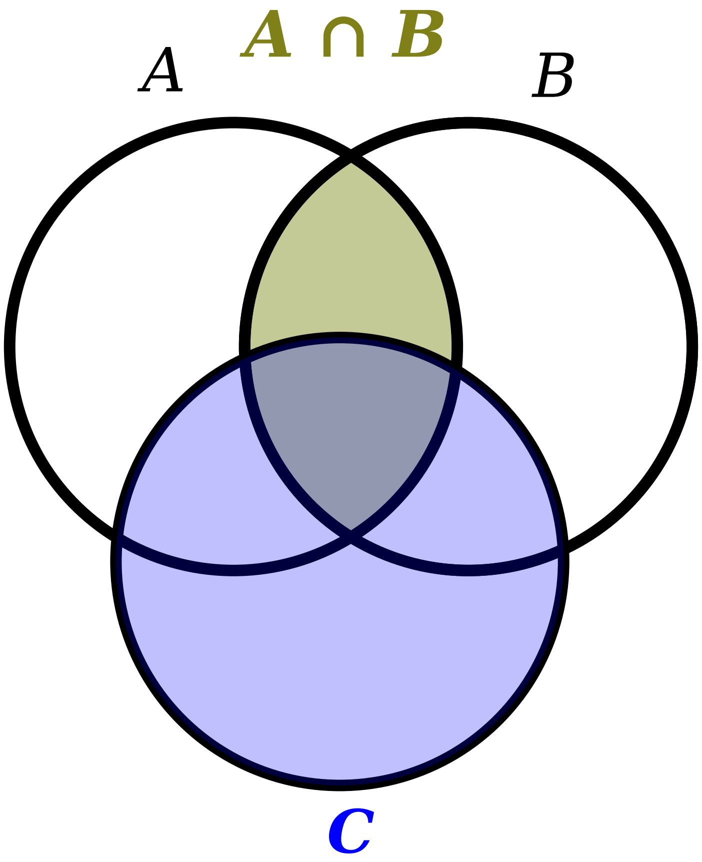

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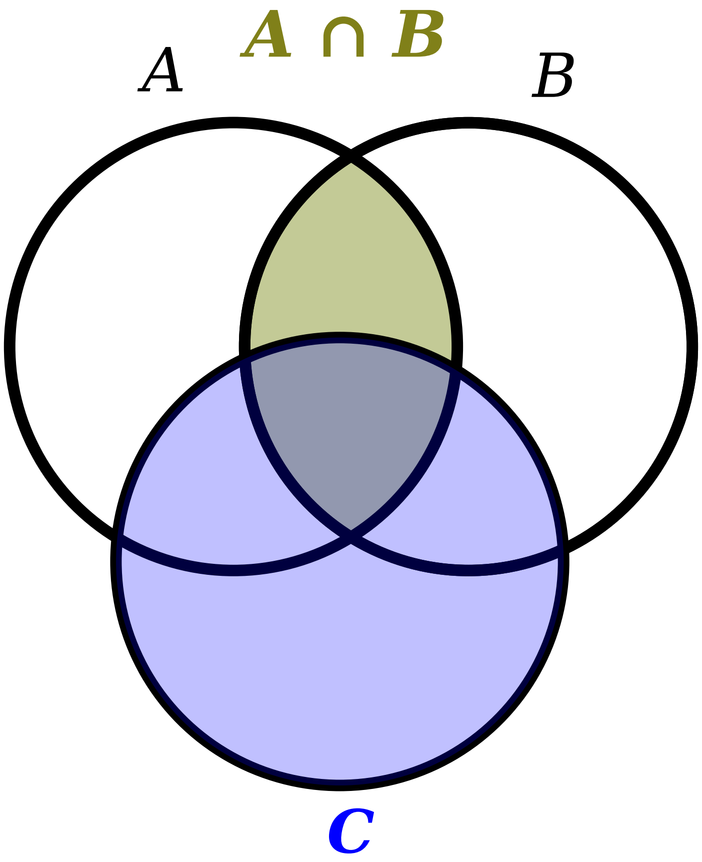

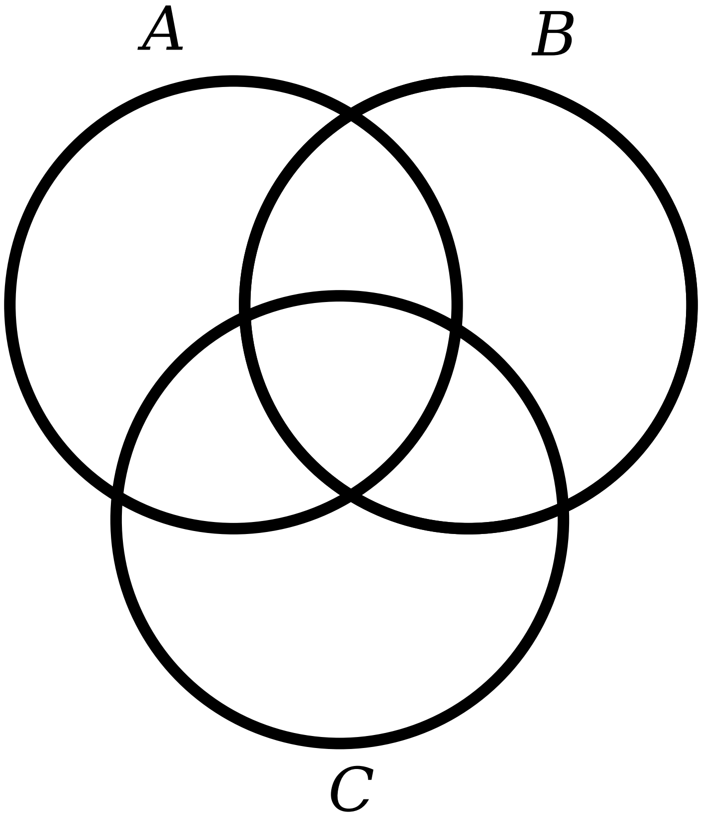

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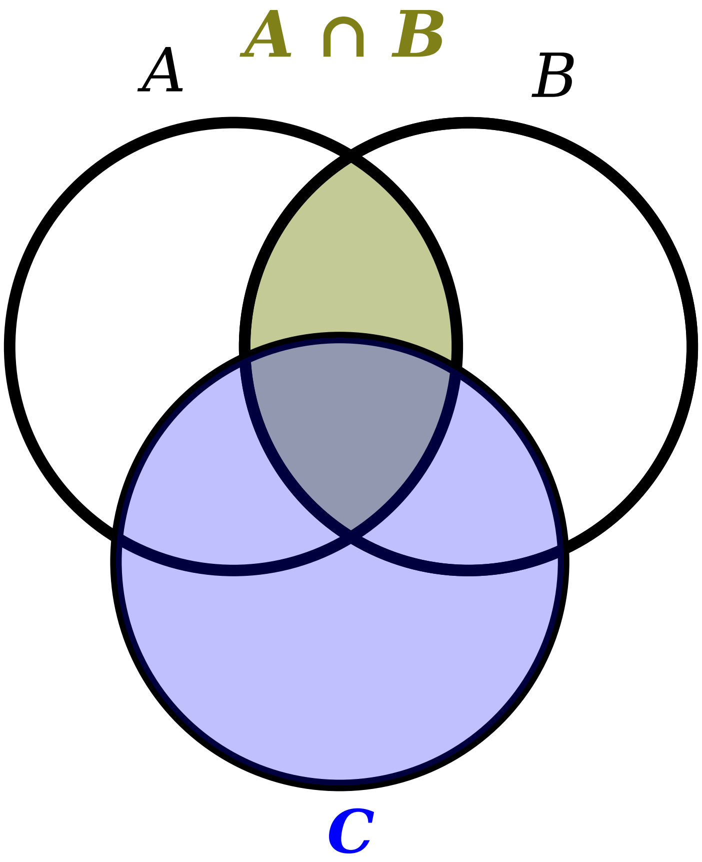

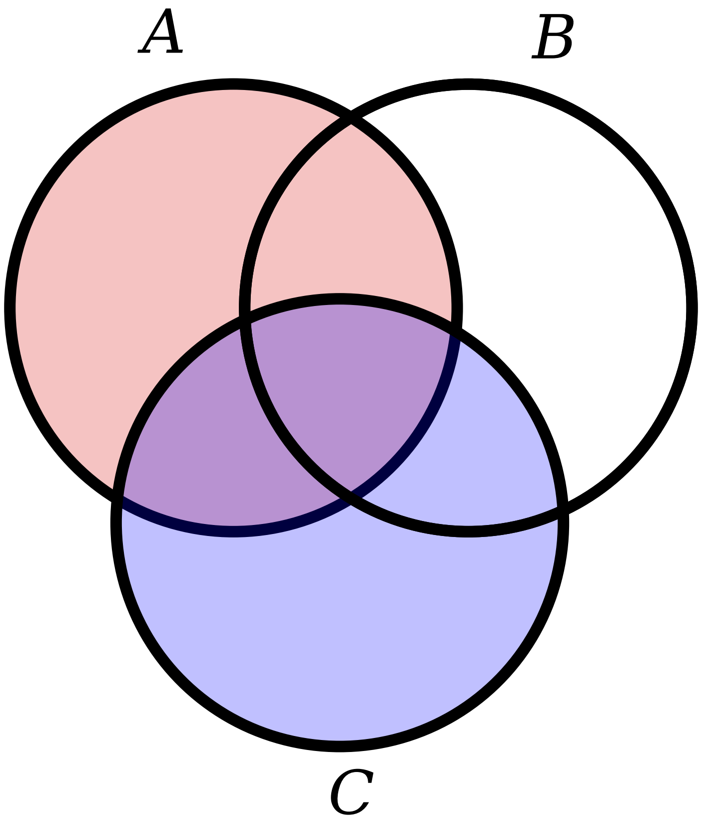

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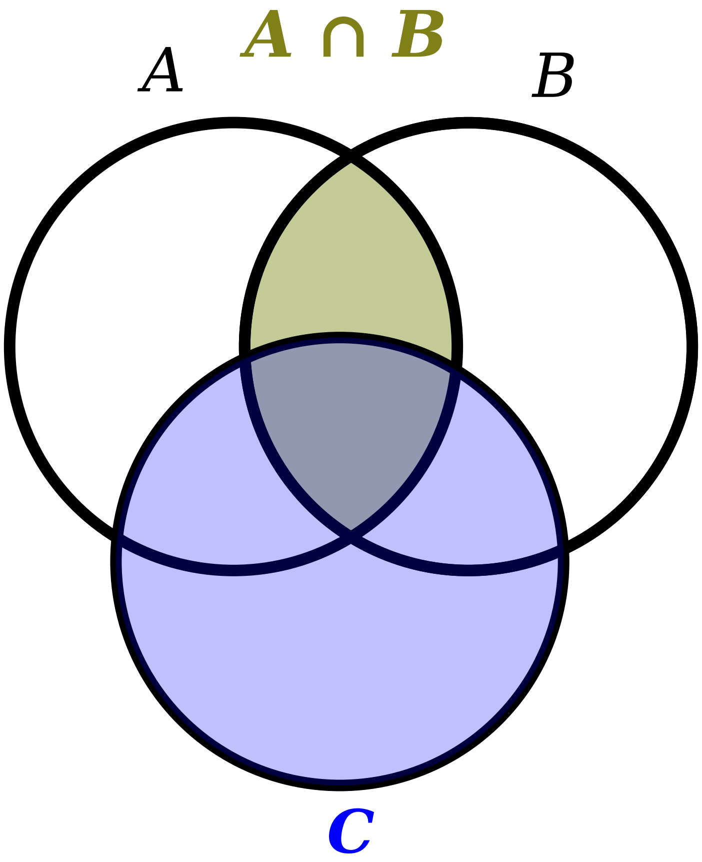

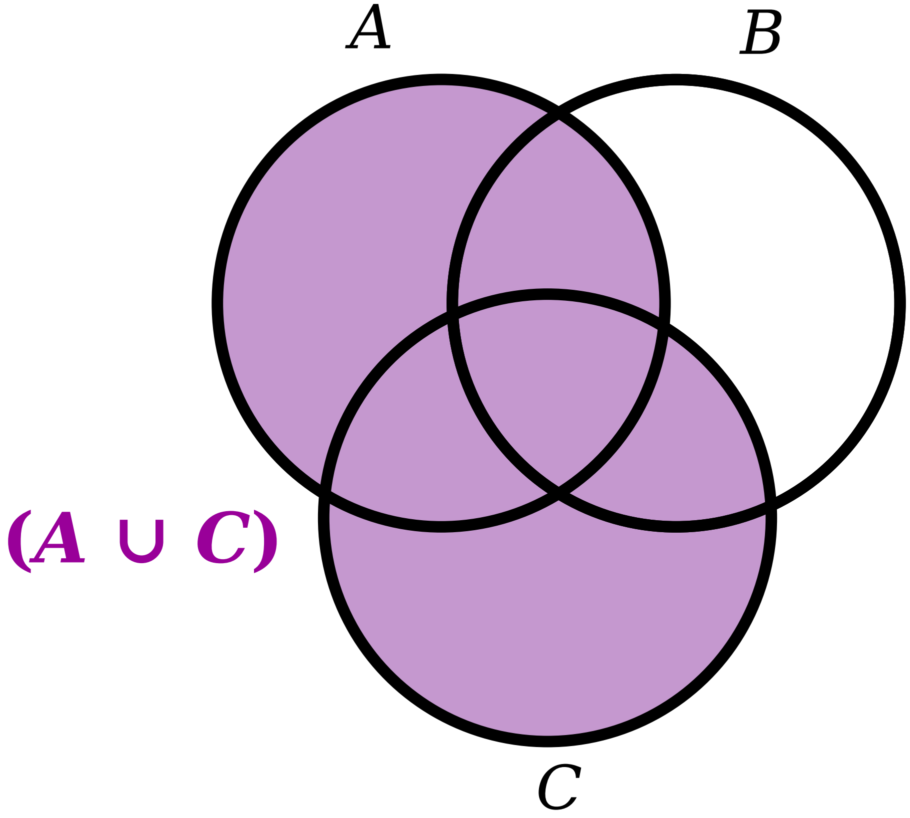

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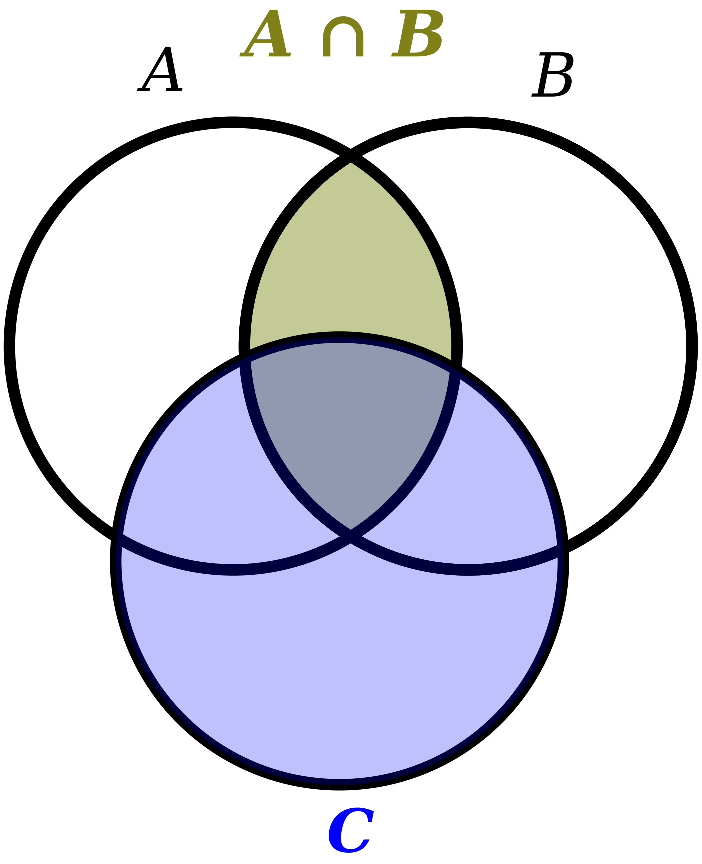

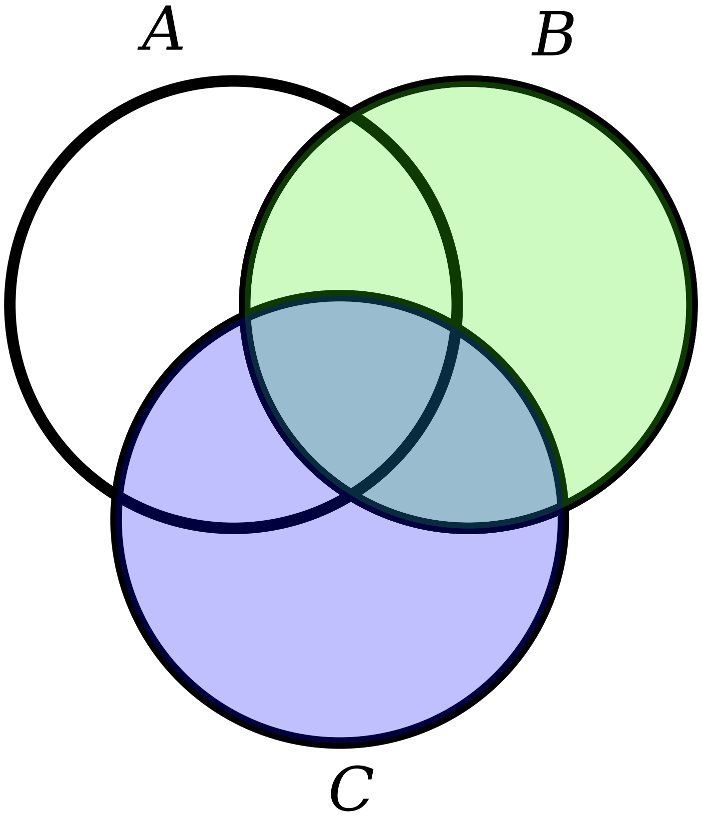

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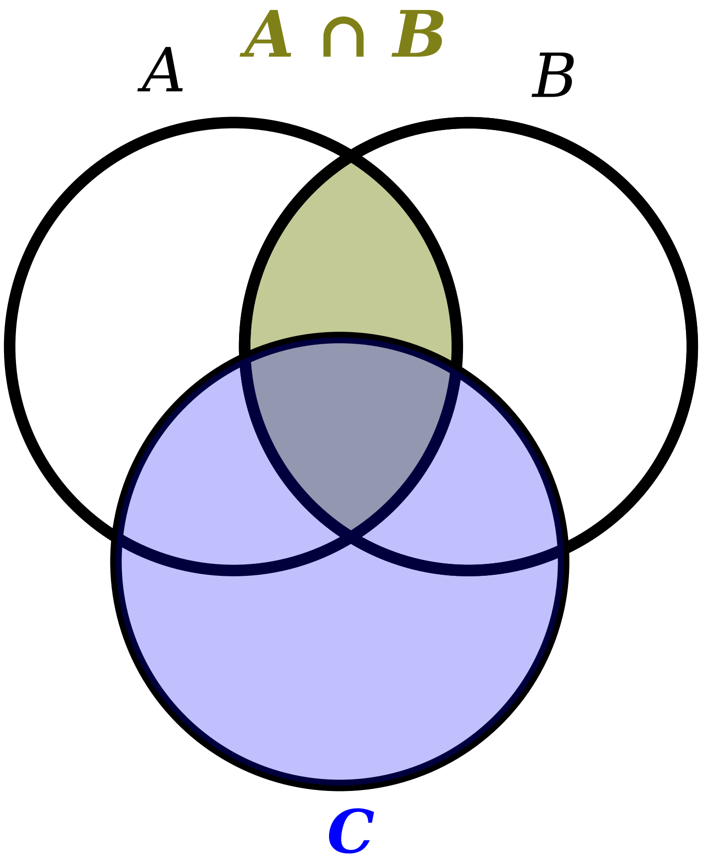

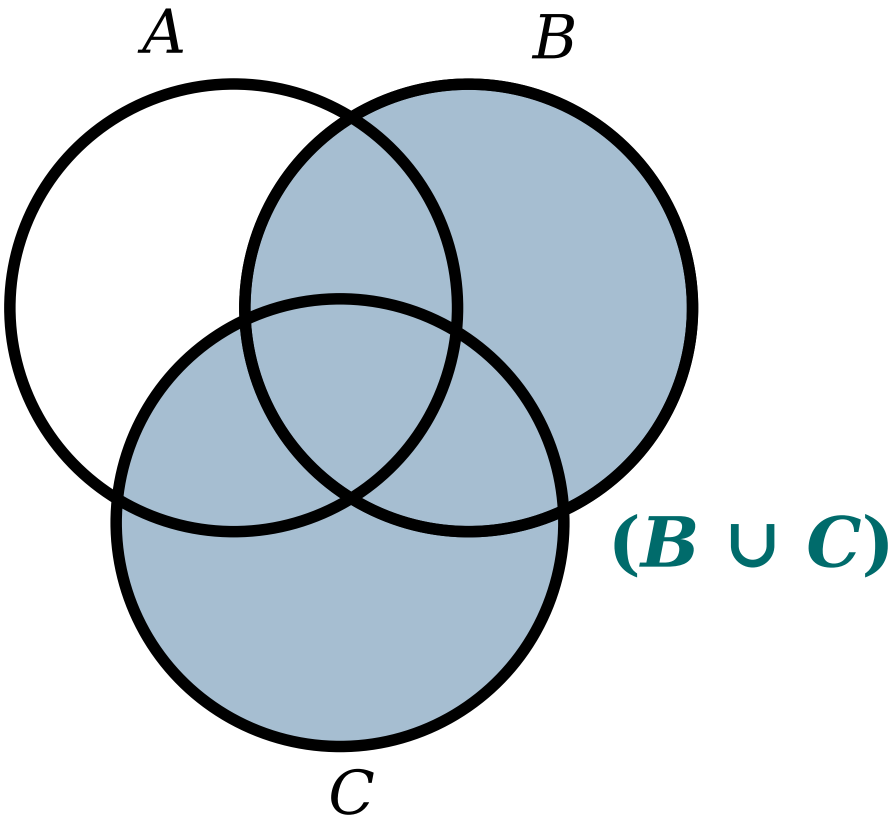

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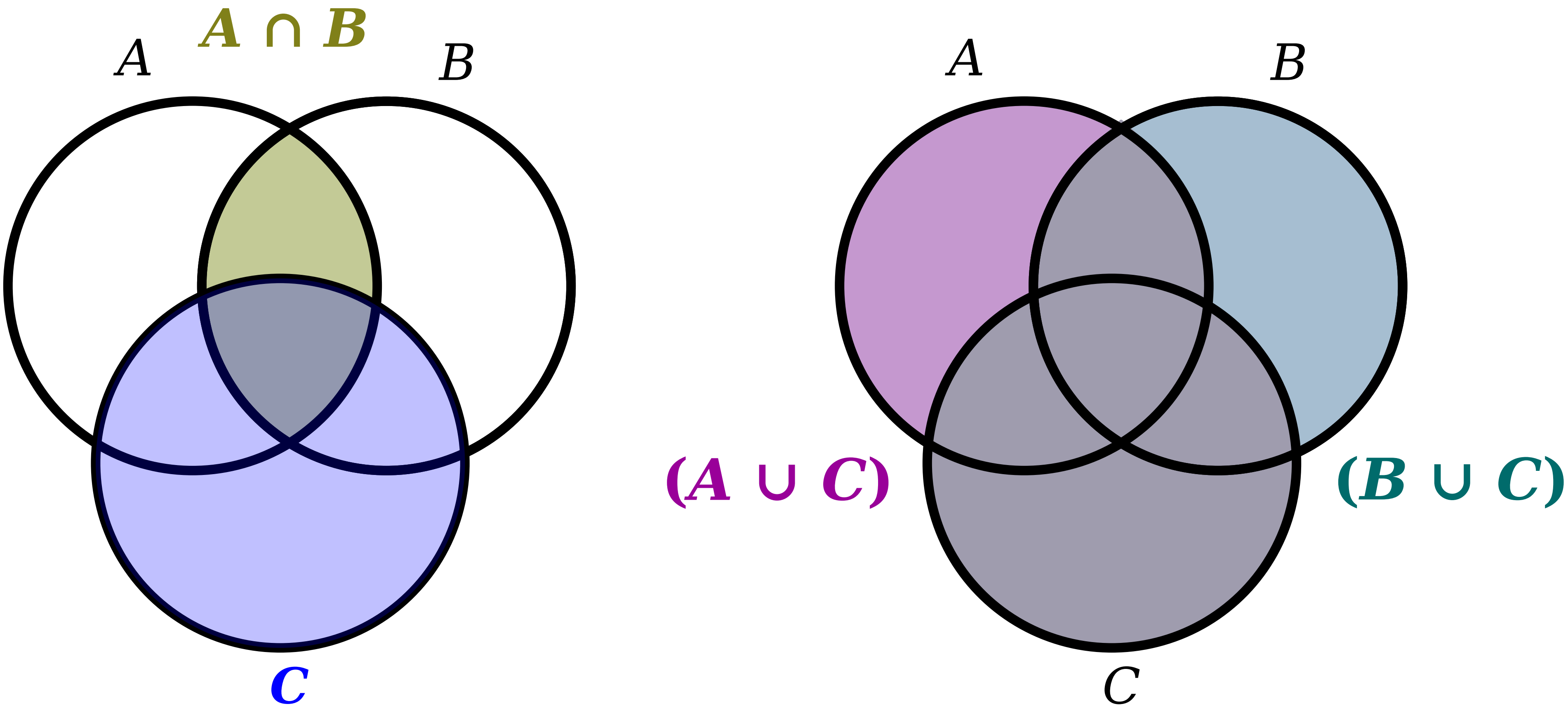

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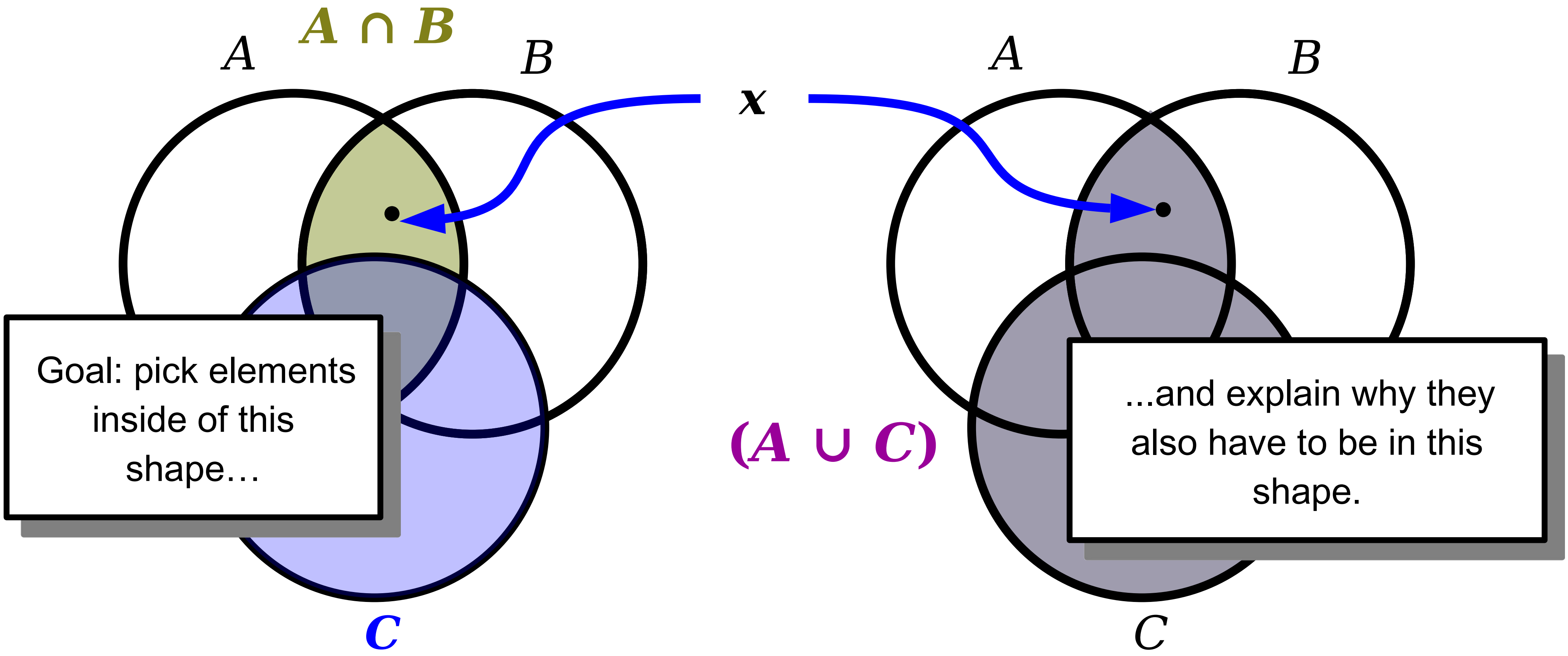

Goal: pick elements inside of this shape…

...and explain why they also have to be in this shape.

Theorem: If A, B, and C are sets, then for any $x \in (A \cap B) \cup C$, we have $x \in (A \cup C) \cap (B \cup C)$.

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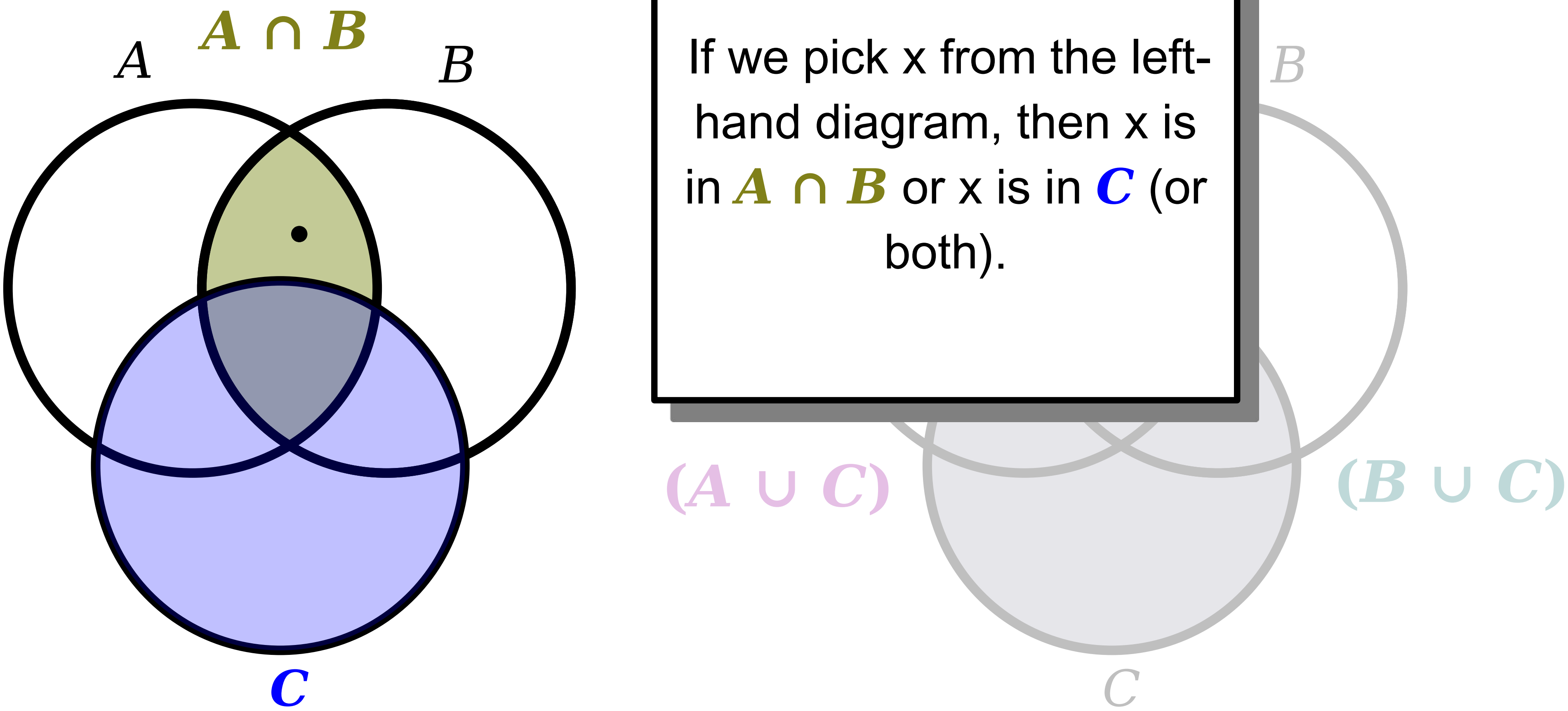
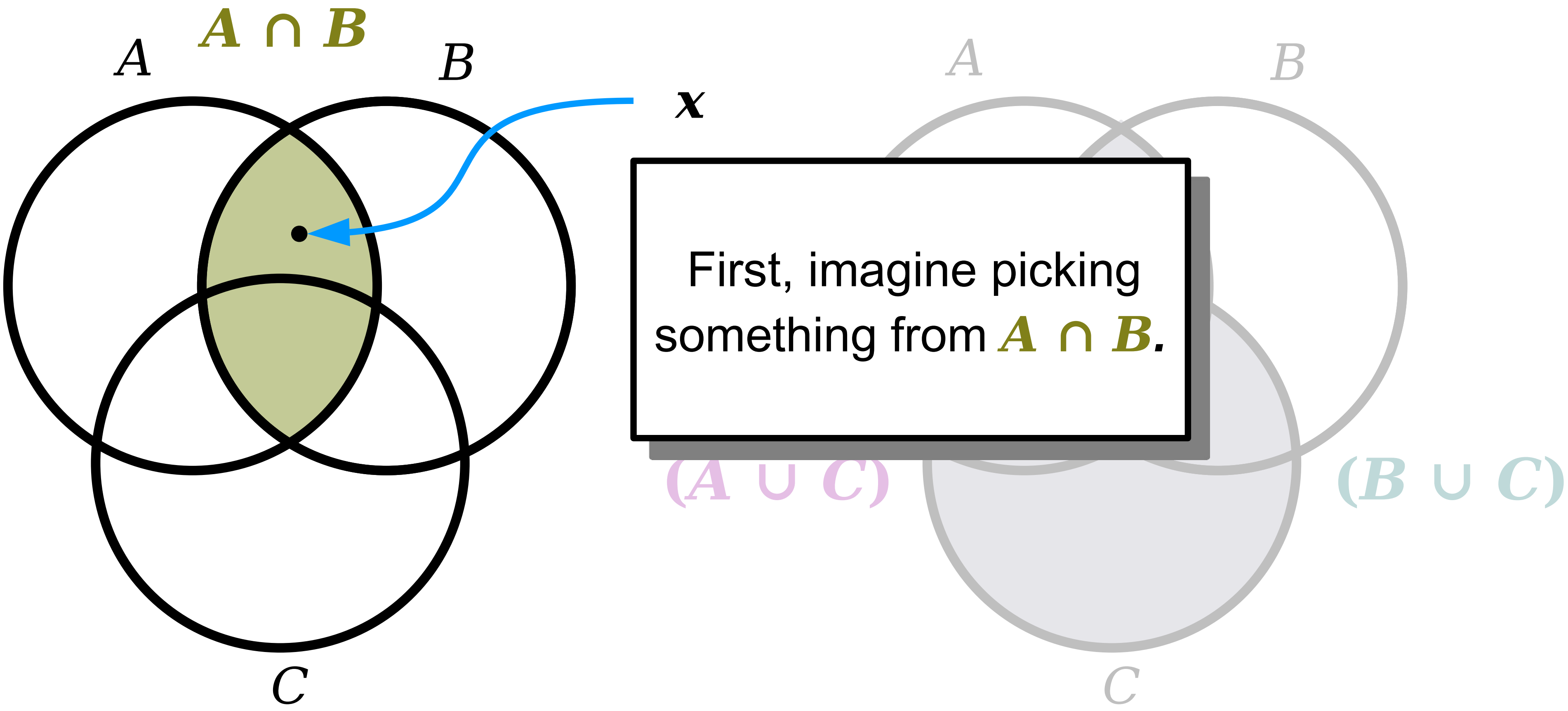


If we pick x from the left-hand diagram, then x is in $A \cap B$ or x is in C (or both).

Theorem: If A, B, and C are sets, then for any $x \in (A \cap B) \cup C$, we have $x \in (A \cup C) \cap (B \cup C)$.

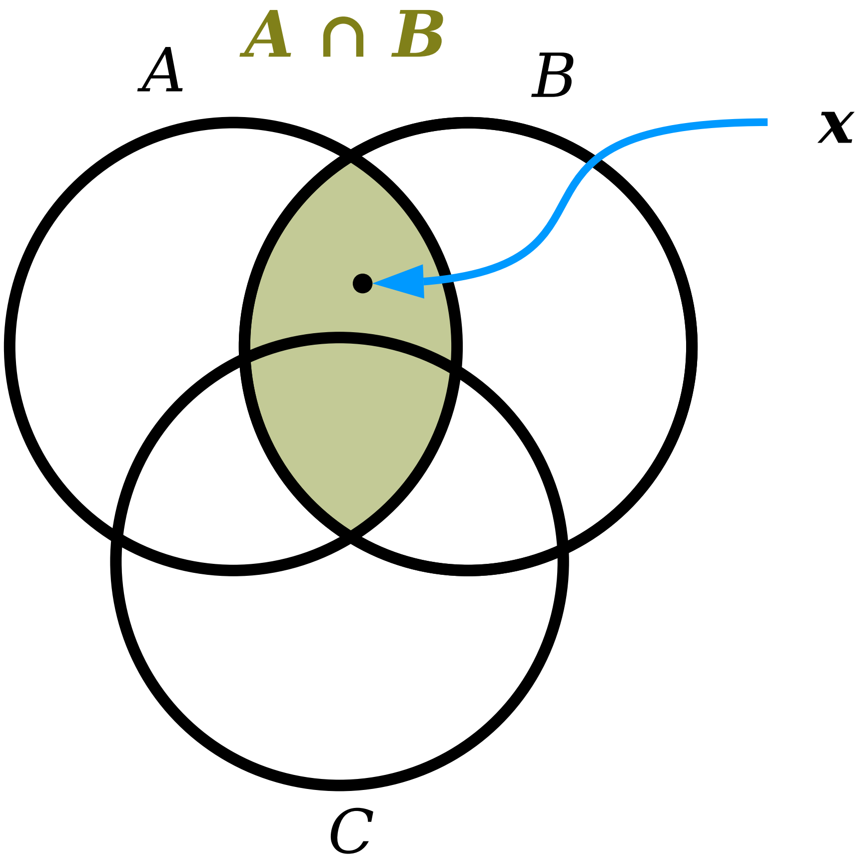
Let’s Draw Some Pictures!

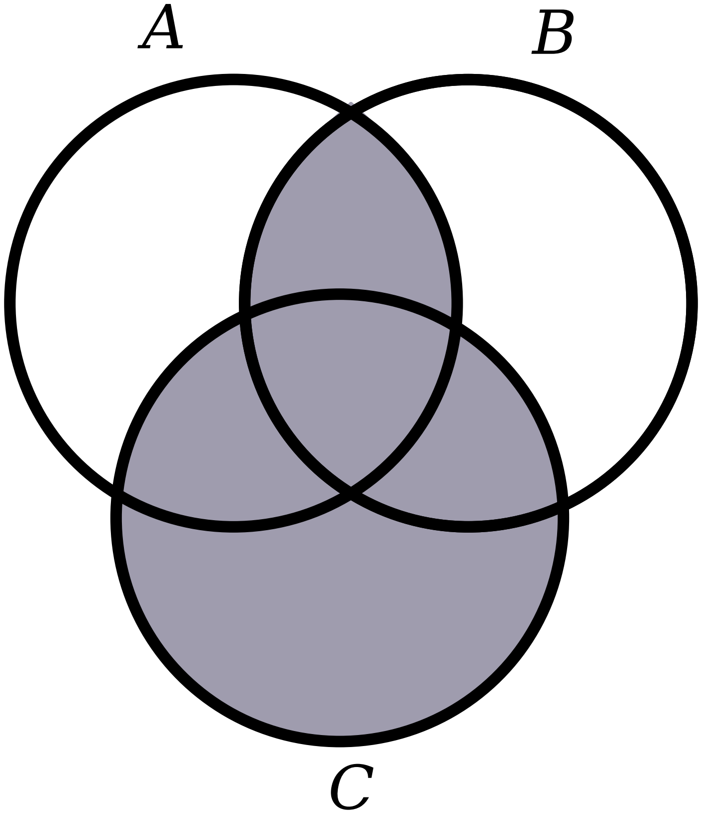


First, imagine picking something from $A \cap B$.

Theorem: If A, B, and C are sets, then for any $x \in (A \cap B) \cup C$, we have $x \in (A \cup C) \cap (B \cup C)$.

Let’s Draw Some Pictures!





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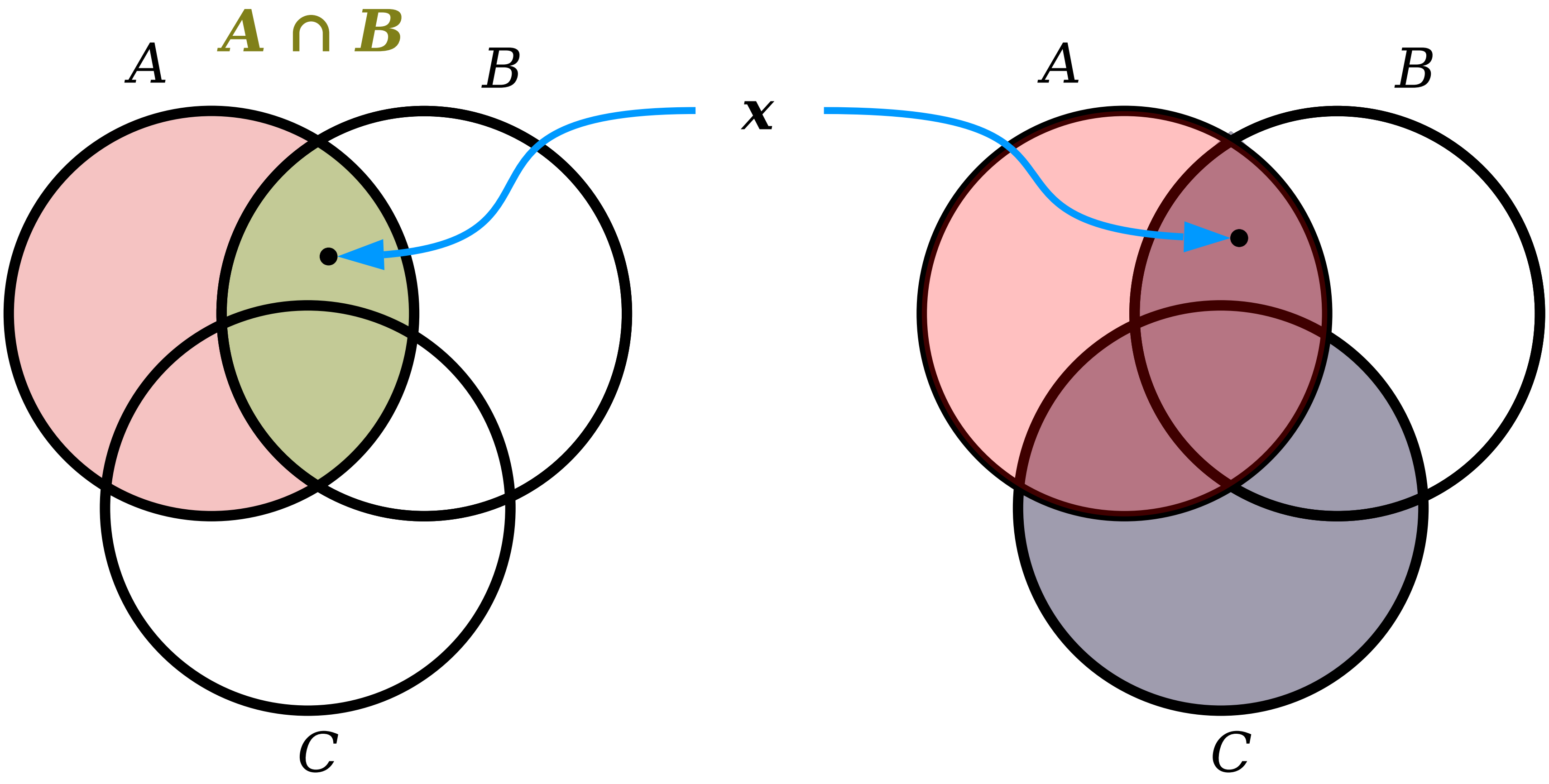

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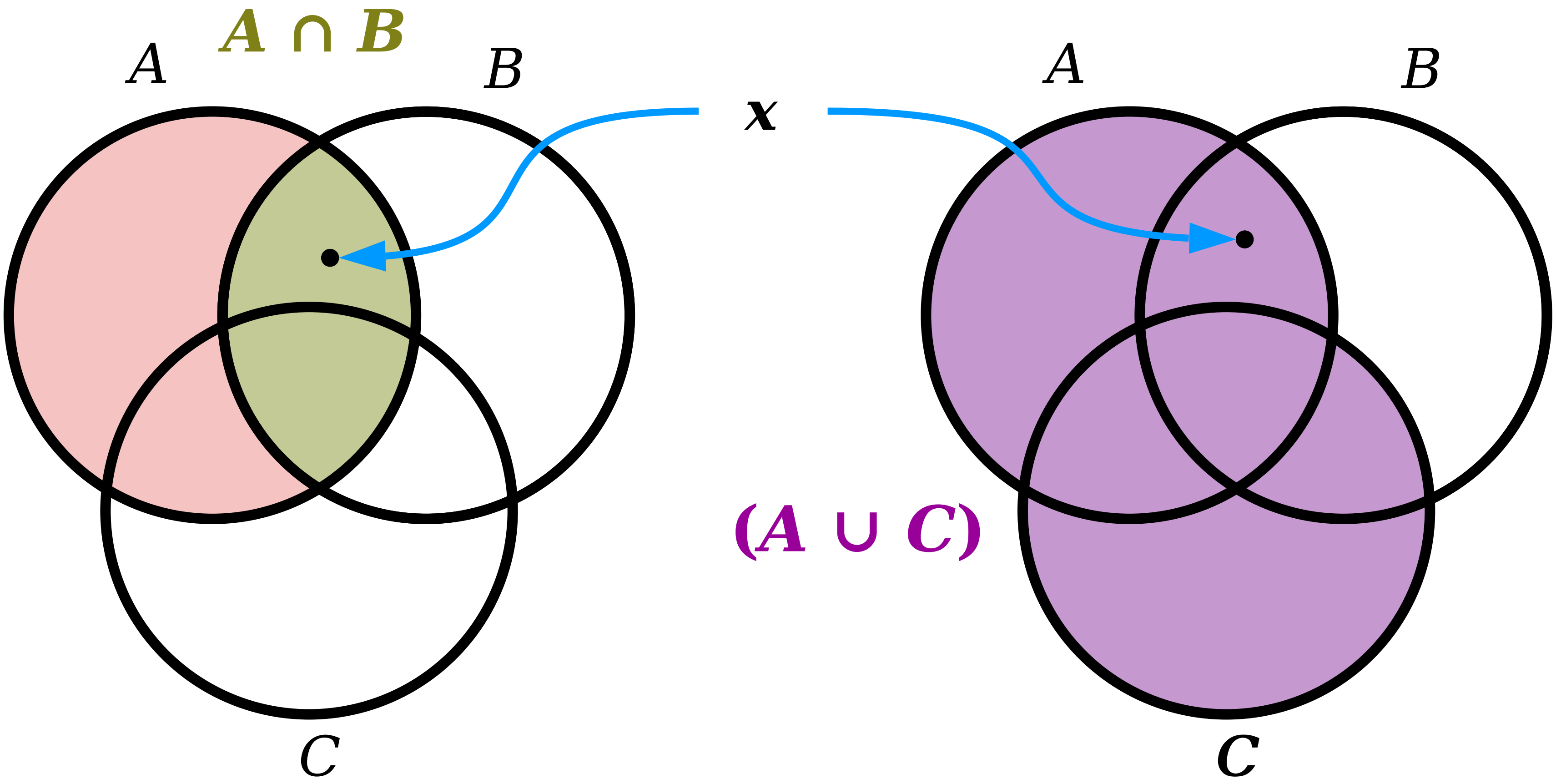

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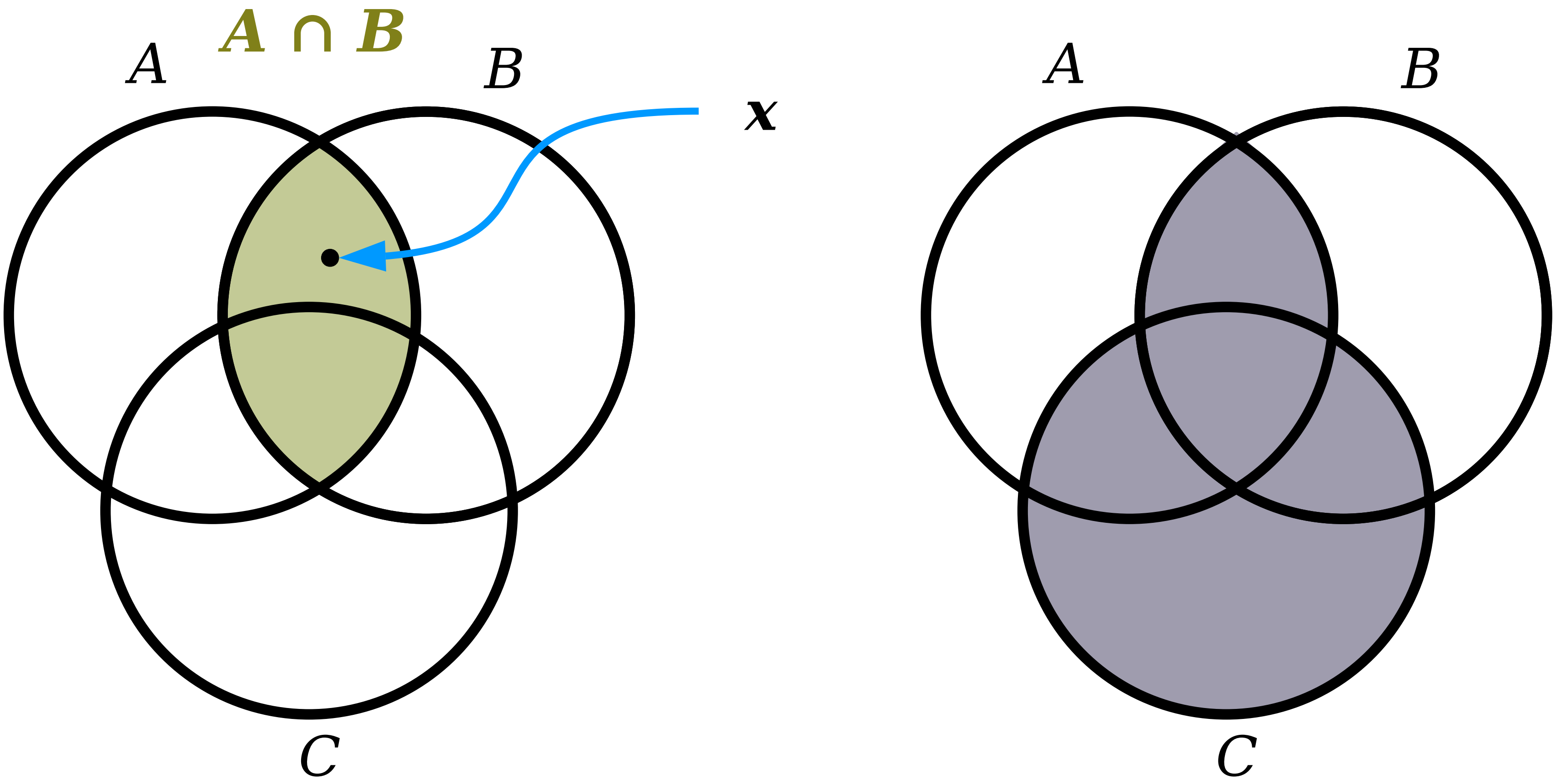

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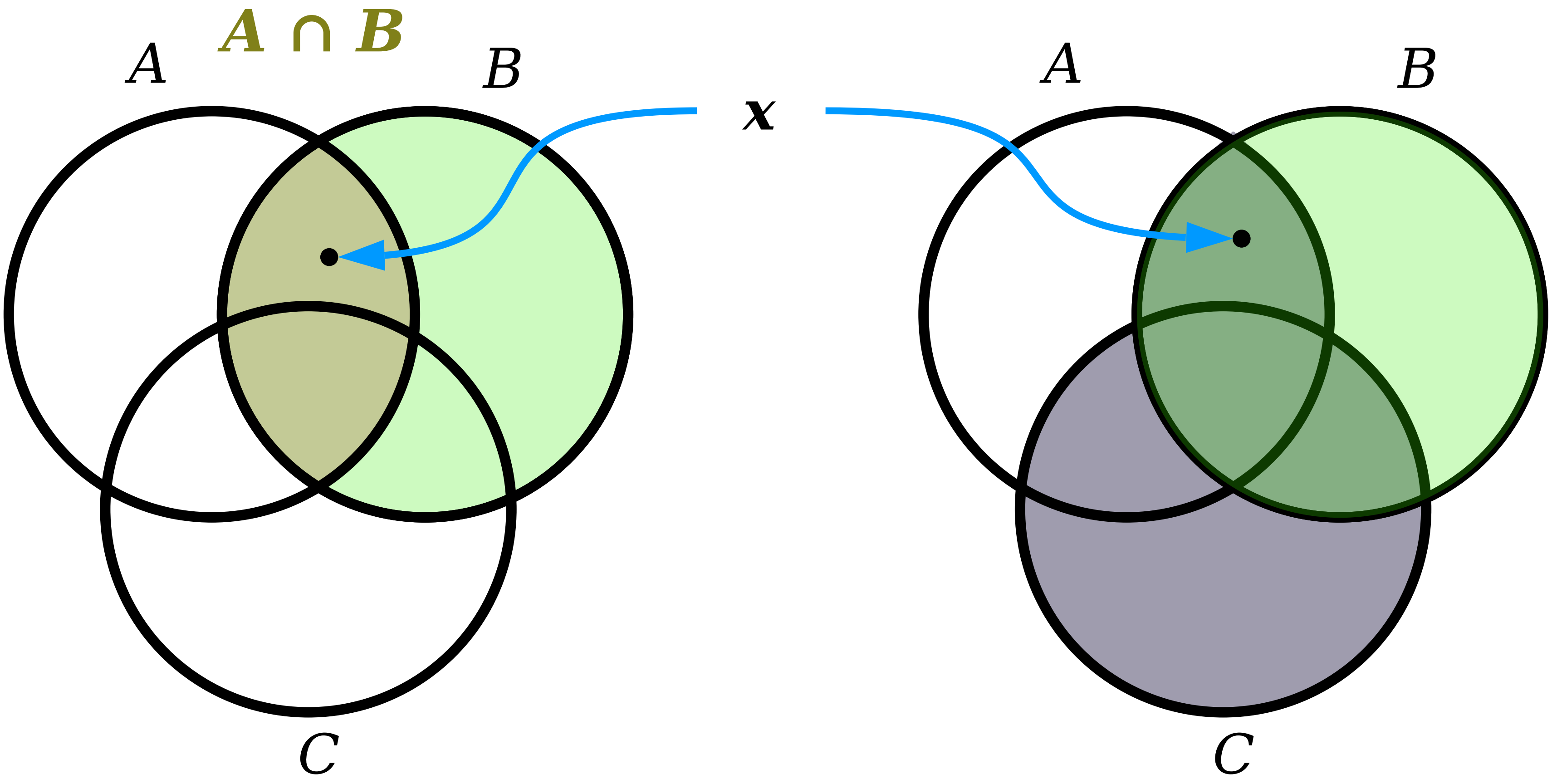

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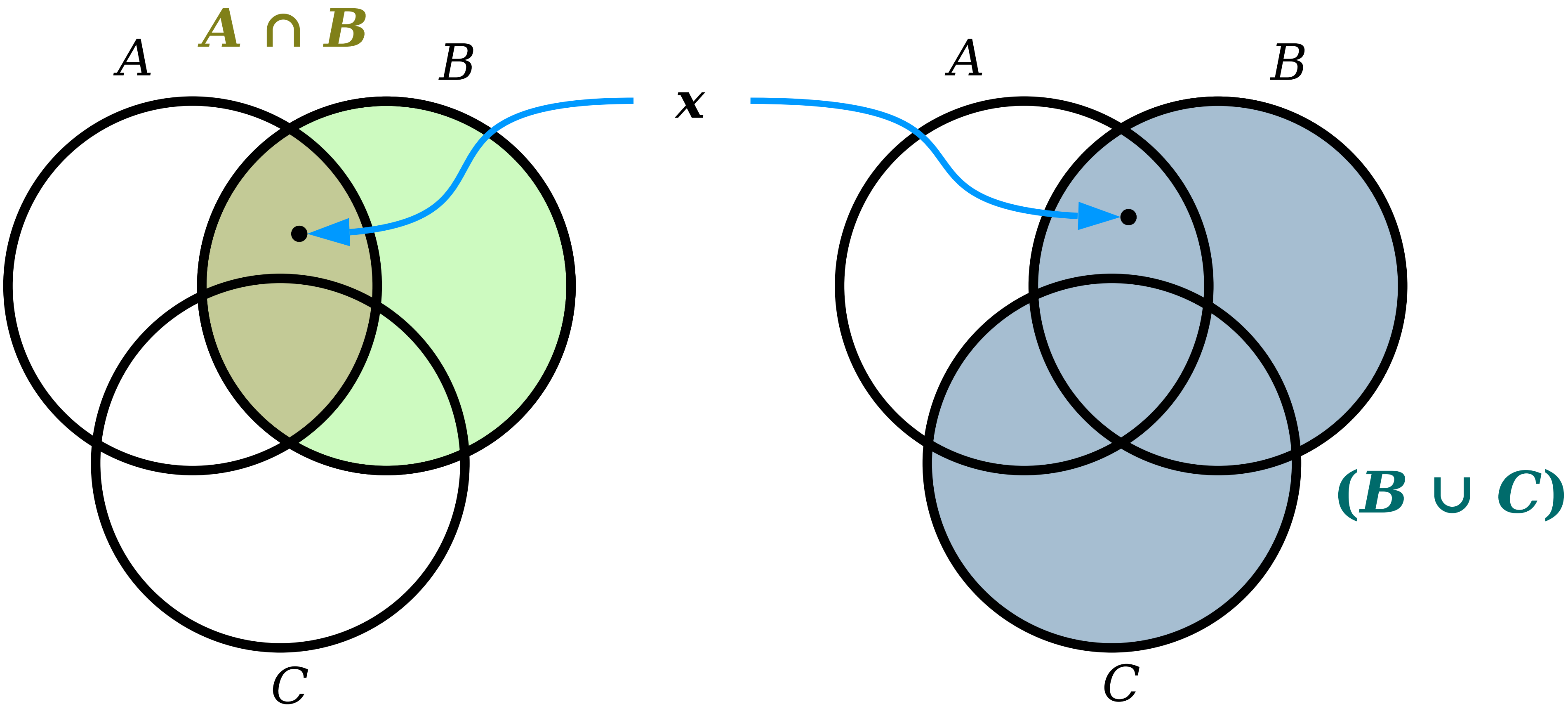

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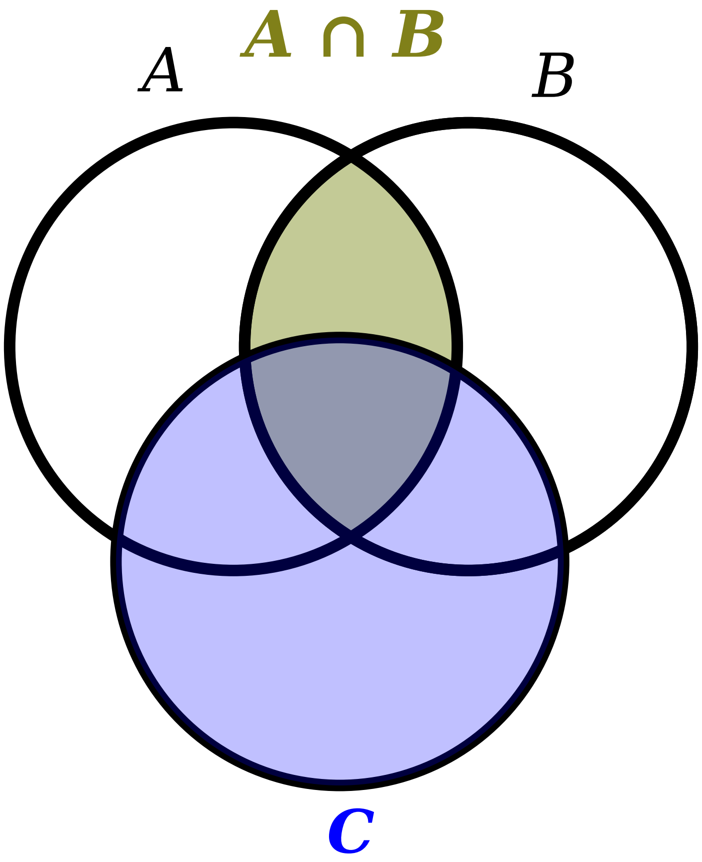

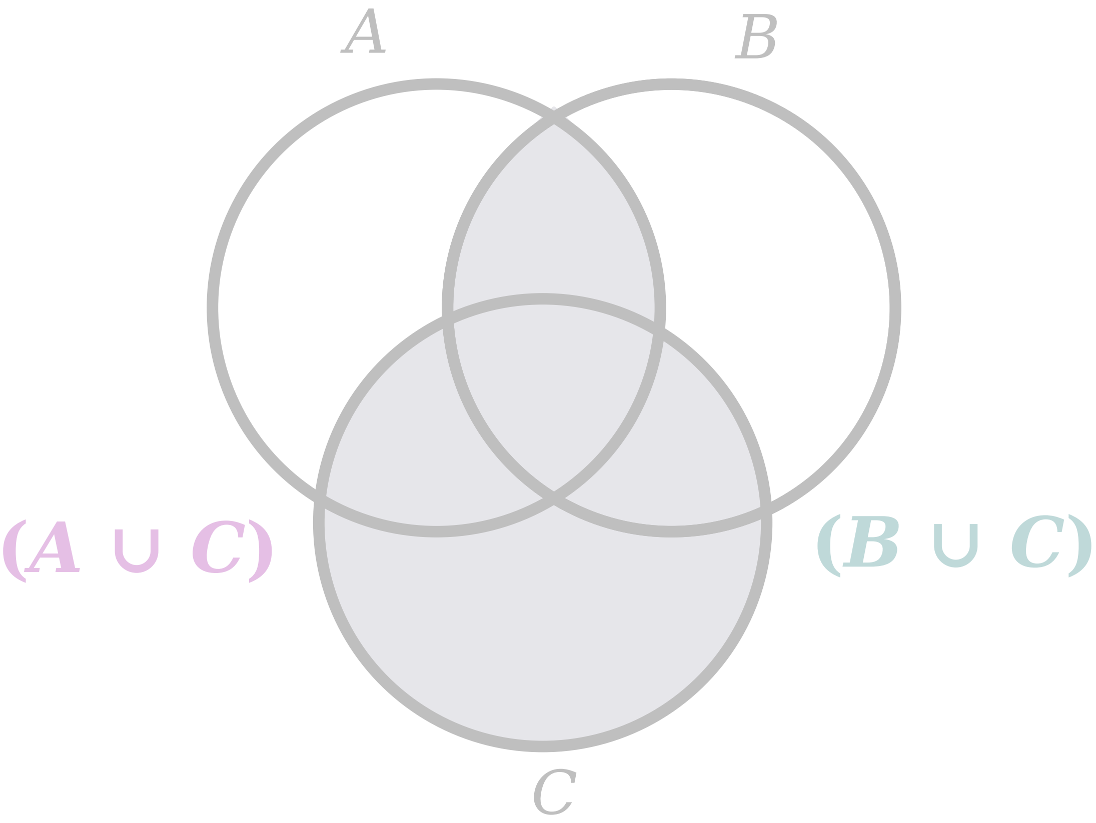

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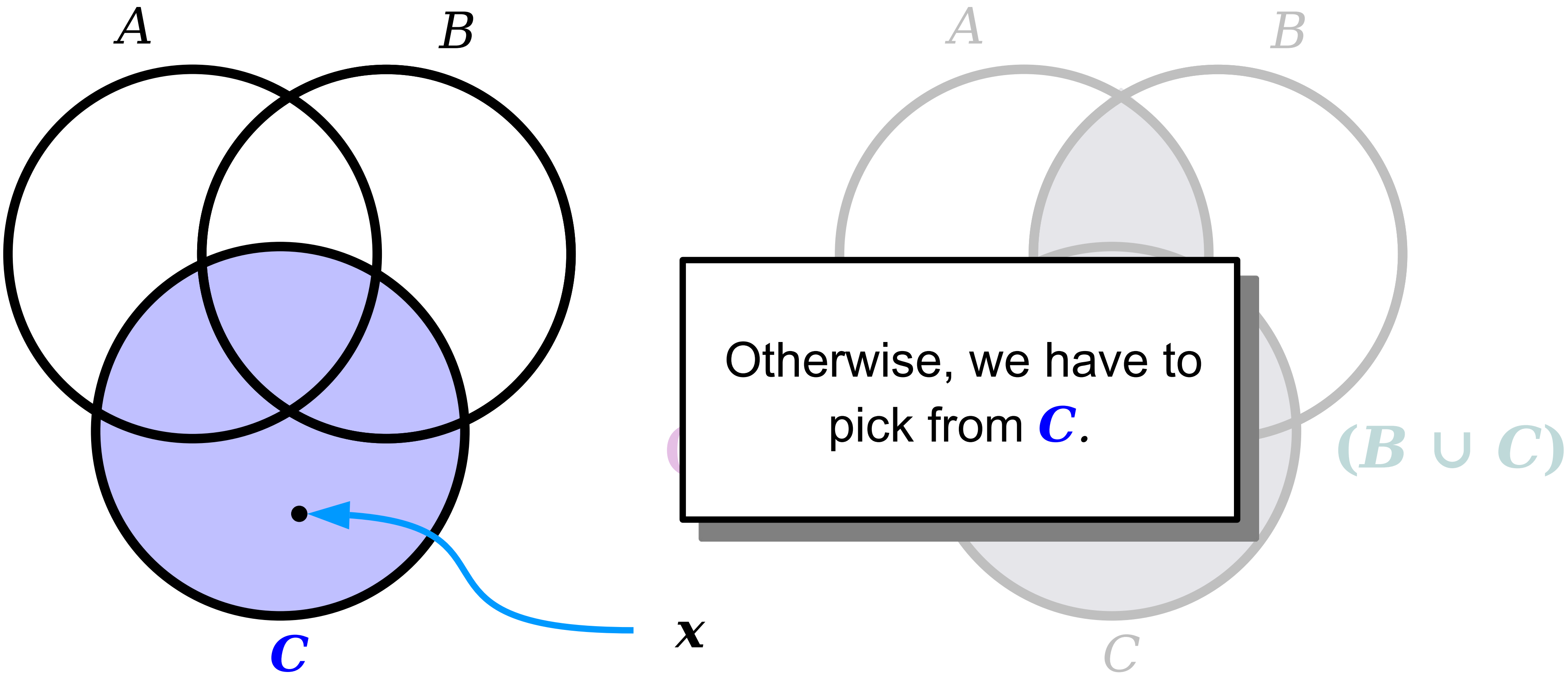

Otherwise, we have to pick from ***C***.

Theorem: If A, B, and C are sets, then for any $x \in (A \cap B) \cup C$, we have $x \in (A \cup C) \cap (B \cup C)$.

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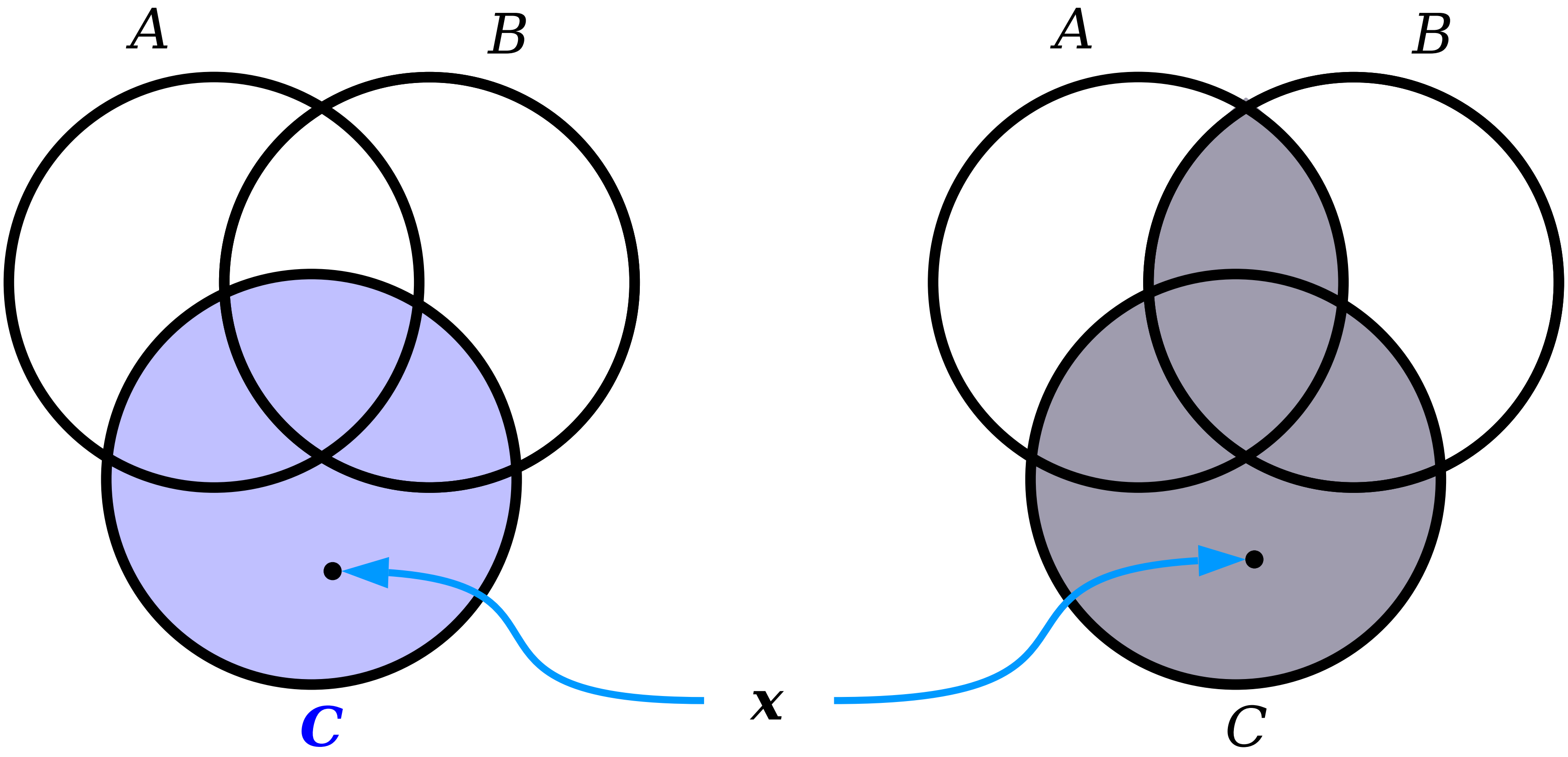

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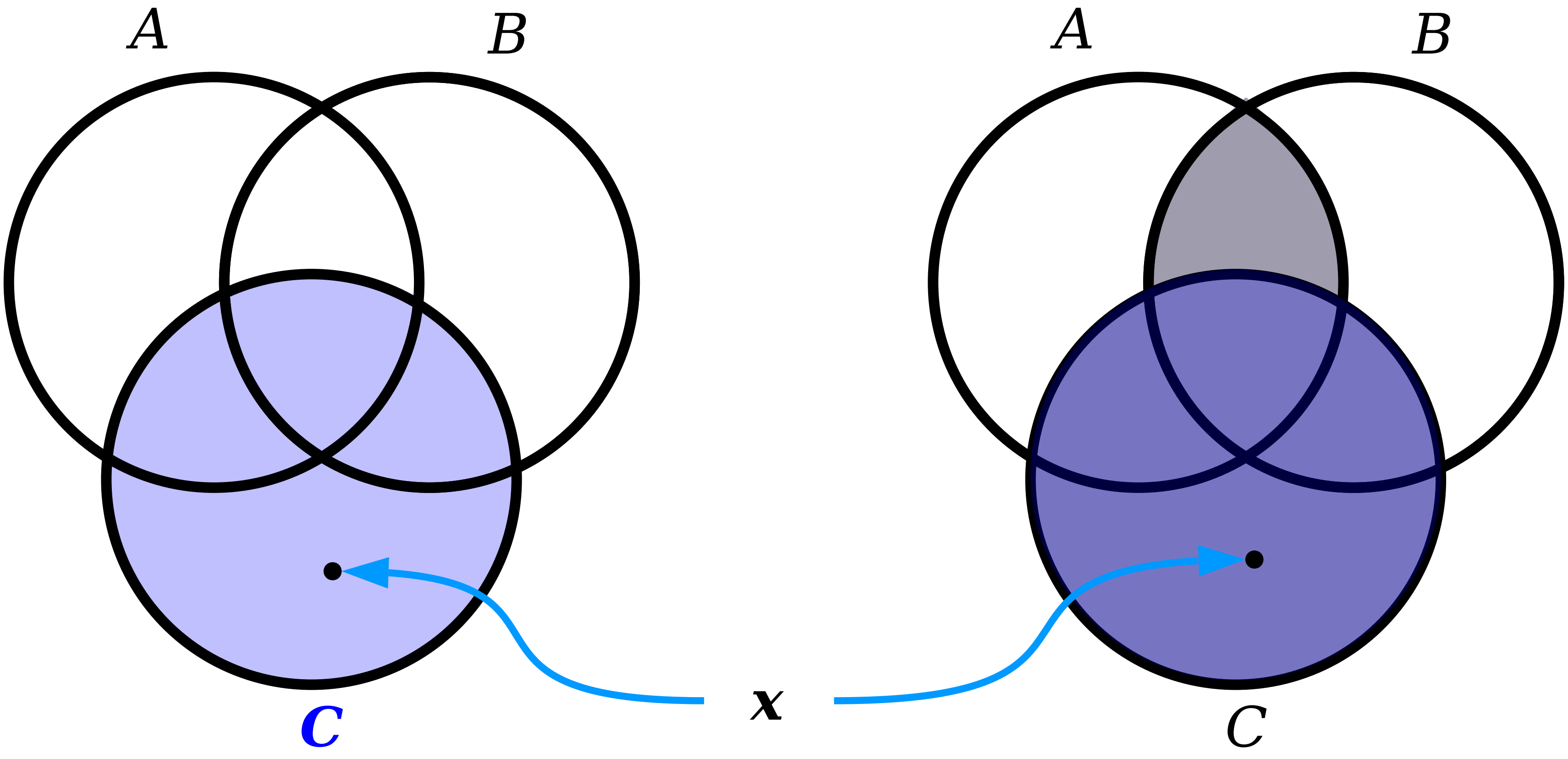

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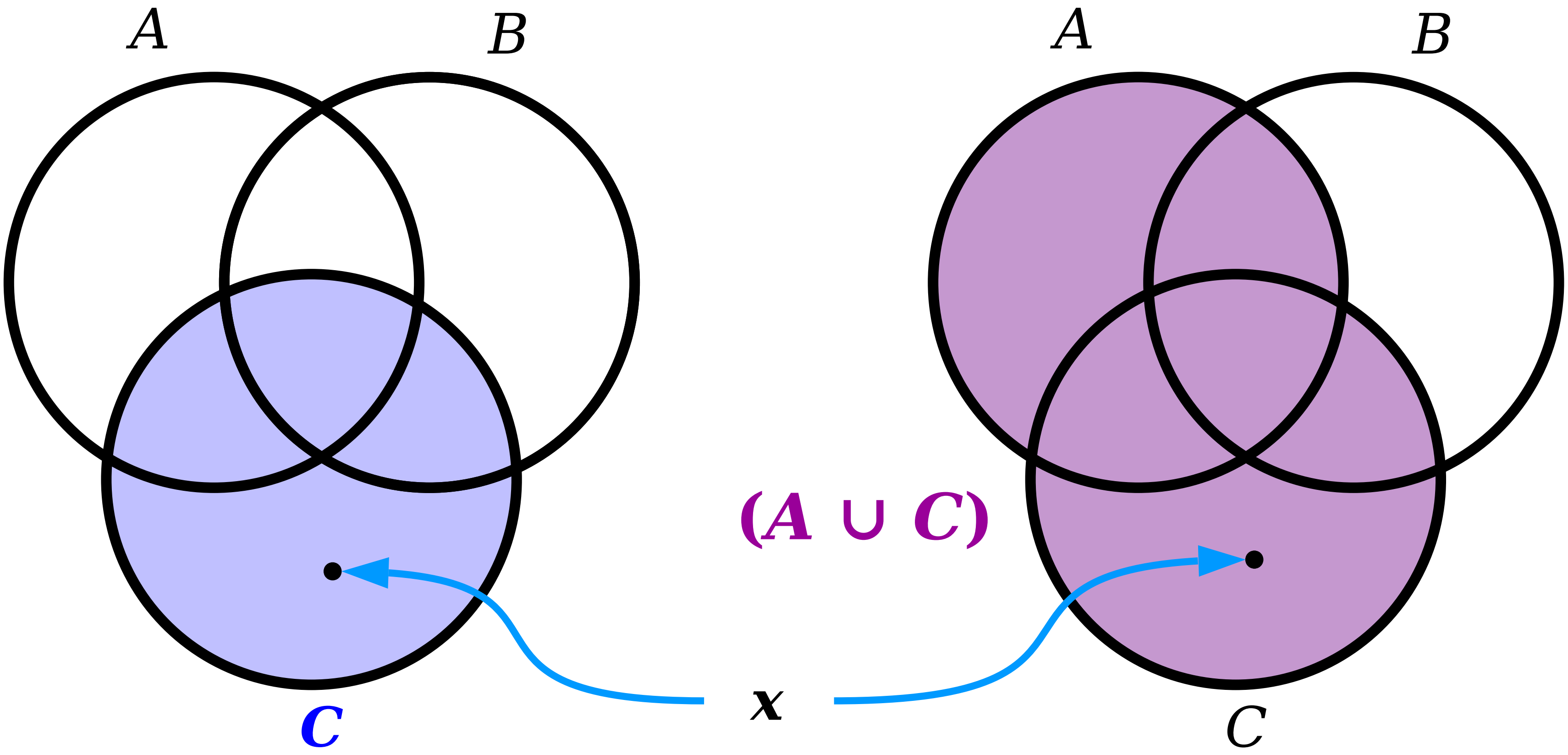

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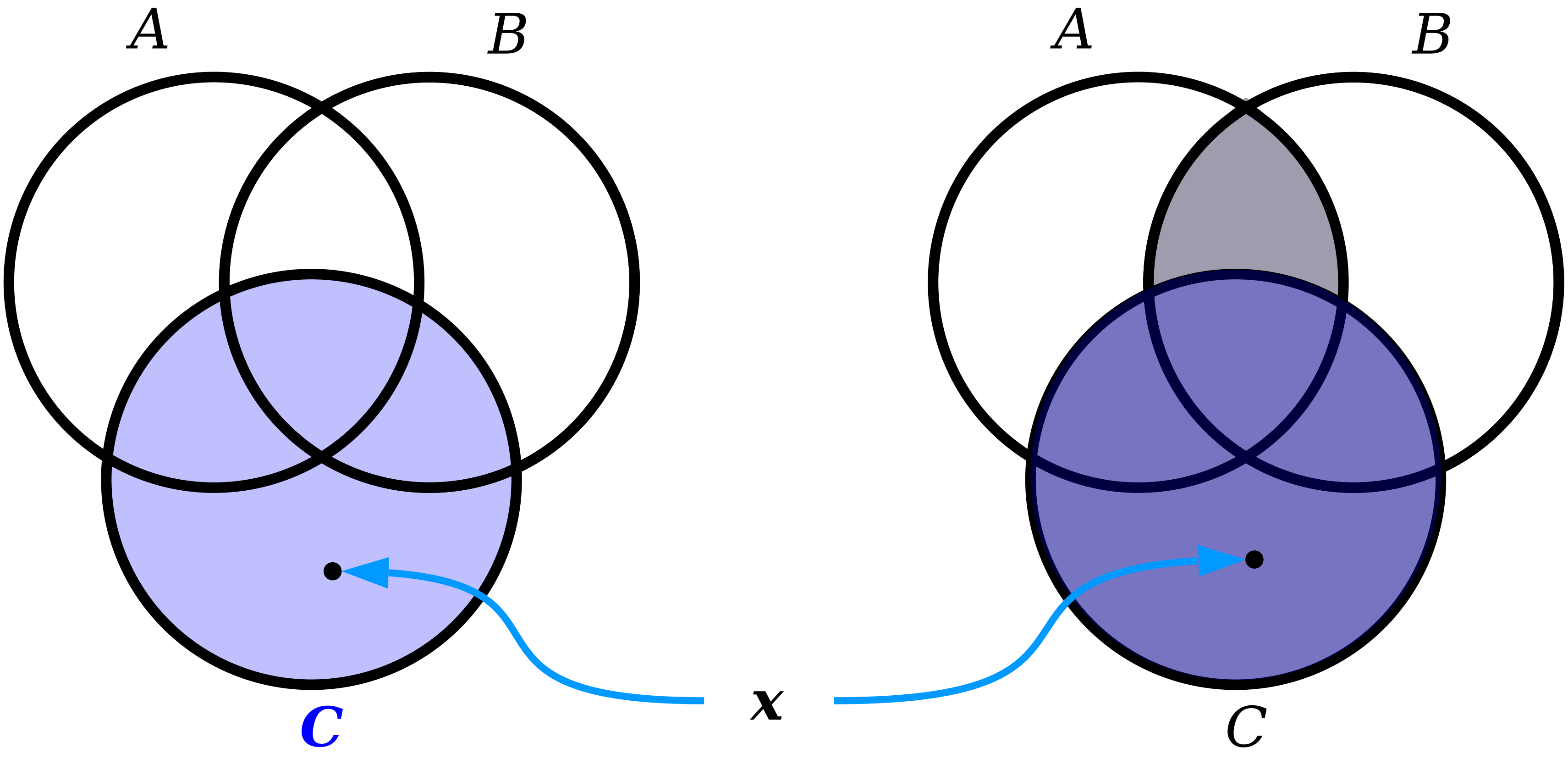

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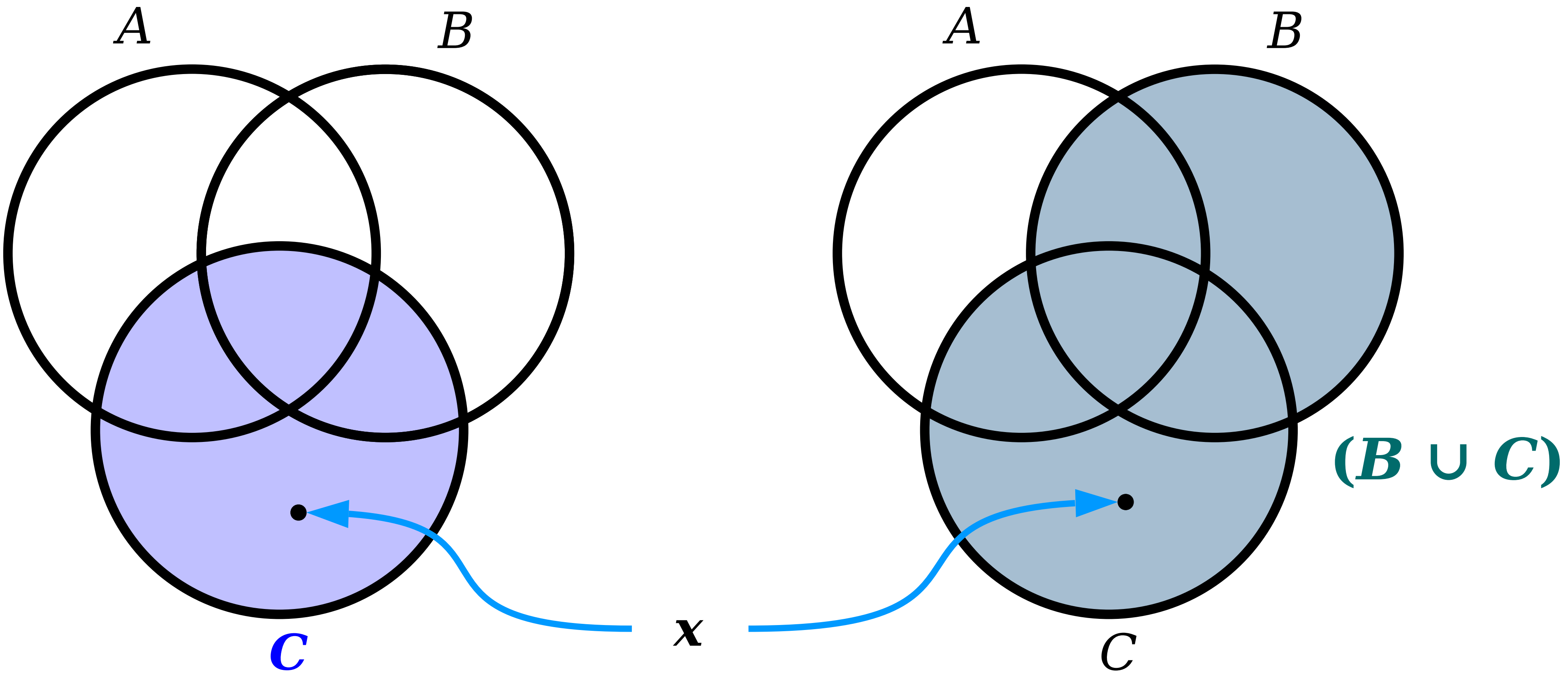

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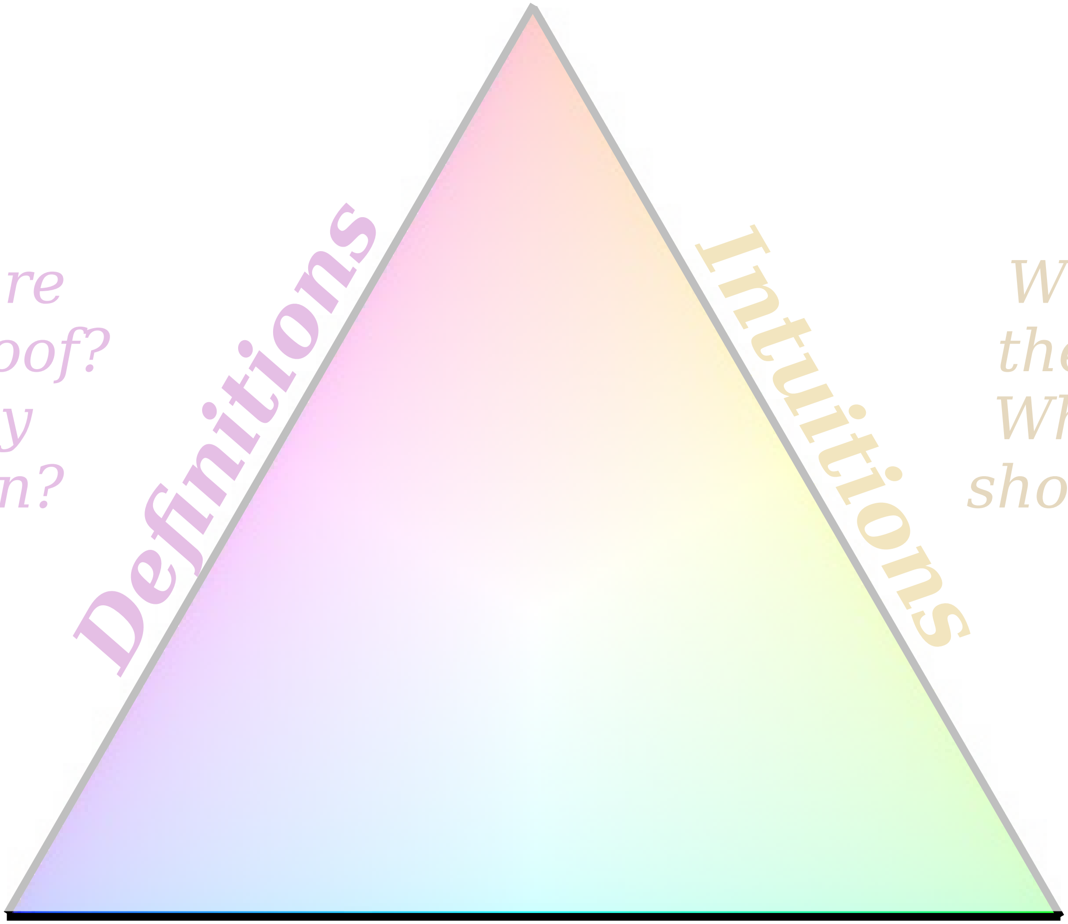

*What terms are
used in this proof?
What do they
formally mean?*

Definitions

Intuitions

*What does this
theorem mean?
Why, intuitively,
should it be true?*

Conventions

*What is the standard
format for writing a proof?
What are the techniques
for doing so?*

Theorem: If A, B, and C are sets, then for any $x \in (A \cap B) \cup C$, we have $x \in (A \cup C) \cap (B \cup C)$.

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Proof: Consider arbitrary sets A, B, and C, then choose any $x \in (A \cap B) \cup C$. We will prove $x \in (A \cup C) \cap (B \cup C)$.

Since $x \in (A \cap B) \cup C$, we know that $x \in A \cap B$ or that $x \in C$. We consider each case separately.

Case 1: $x \in C$.

Case 2: $x \in A \cap B$.

Theorem: If A, B, and C are sets, then for any $x \in (A \cap B) \cup C$, we have $x \in (A \cup C) \cap (B \cup C)$.

Proof: Consider arbitrary sets A, B, and C, then choose any $x \in (A \cap B) \cup C$. We will prove $x \in (A \cup C) \cap (B \cup C)$.

Since $x \in (A \cap B) \cup C$, we know that $x \in A \cap B$ or that $x \in C$. We consider each case separately.

Case 1: $x \in C$. This in turn means that $x \in A \cup C$ and that $x \in B \cup C$.

Case 2: $x \in A \cap B$.

Theorem: If A, B, and C are sets, then for any $x \in (A \cap B) \cup C$, we have $x \in (A \cup C) \cap (B \cup C)$.

Proof: Consider arbitrary sets A, B, and C, then choose any $x \in (A \cap B) \cup C$. We will prove $x \in (A \cup C) \cap (B \cup C)$.

Since $x \in (A \cap B) \cup C$, we know that $x \in A \cap B$ or that $x \in C$. We consider each case separately.

Case 1: x ∈ C. This in turn means that $x \in A \cup C$ and that $x \in B \cup C$.

Case 2: x ∈ A ∩ B. From $x \in A \cap B$, we learn that $x \in A$ and that $x \in B$.

Theorem: If A, B, and C are sets, then for any $x \in (A \cap B) \cup C$, we have $x \in (A \cup C) \cap (B \cup C)$.

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Since $x \in (A \cap B) \cup C$, we know that $x \in A \cap B$ or that $x \in C$. We consider each case separately.

Case 1: $x \in C$. This in turn means that $x \in A \cup C$ and that $x \in B \cup C$.

Case 2: $x \in A \cap B$. From $x \in A \cap B$, we learn that $x \in A$ and that $x \in B$. Therefore, we know that $x \in A \cup C$ and that $x \in B \cup C$.

Theorem: If A, B, and C are sets, then for any $x \in (A \cap B) \cup C$, we have $x \in (A \cup C) \cap (B \cup C)$.

Proof: Consider arbitrary sets A, B, and C, then choose any $x \in (A \cap B) \cup C$. We will prove $x \in (A \cup C) \cap (B \cup C)$.

Since $x \in (A \cap B) \cup C$, we know that $x \in A \cap B$ or that $x \in C$. We consider each case separately.

Case 1: $x \in C$. This in turn means that $x \in A \cup C$ and that $x \in B \cup C$.

Case 2: $x \in A \cap B$. From $x \in A \cap B$, we learn that $x \in A$ and that $x \in B$. Therefore, we know that $x \in A \cup C$ and that $x \in B \cup C$.

In either case, we learn that $x \in A \cup C$ and $x \in B \cup C$.

Theorem: If A, B, and C are sets, then for any $x \in (A \cap B) \cup C$, we have $x \in (A \cup C) \cap (B \cup C)$.

Proof: Consider arbitrary sets A, B, and C, then choose any $x \in (A \cap B) \cup C$. We will prove $x \in (A \cup C) \cap (B \cup C)$.

Since $x \in (A \cap B) \cup C$, we know that $x \in A \cap B$ or that $x \in C$. We consider each case separately.

Case 1: $x \in C$. This in turn means that $x \in A \cup C$ and that $x \in B \cup C$.

Case 2: $x \in A \cap B$. From $x \in A \cap B$, we learn that $x \in A$ and that $x \in B$. Therefore, we know that $x \in A \cup C$ and that $x \in B \cup C$.

In either case, we learn that $x \in A \cup C$ and $x \in B \cup C$. This establishes that $x \in (A \cup C) \cap (B \cup C)$, as required.

Theorem: If A, B, and C are sets, then for any $x \in (A \cap B) \cup C$, we have $x \in (A \cup C) \cap (B \cup C)$.

Proof: Consider arbitrary sets A, B, and C, then choose any $x \in (A \cap B) \cup C$. We will prove $x \in (A \cup C) \cap (B \cup C)$.

Since $x \in (A \cap B) \cup C$, we know that $x \in A \cap B$ or that $x \in C$. We consider each case separately.

Case 1: x ∈ C. This in turn means that $x \in A \cup C$ and that $x \in B \cup C$.

Case 2: x ∈ A ∩ B. From $x \in A \cap B$, we learn that $x \in A$ and that $x \in B$. Therefore, we know that $x \in A \cup C$ and that $x \in B \cup C$.

In either case, we learn that $x \in A \cup C$ and $x \in B \cup C$. This establishes that $x \in (A \cup C) \cap (B \cup C)$, as required. ■

Theorem: If A, B, and C are sets, then for any $x \in (A \cap B) \cup C$, we have $x \in (A \cup C) \cap (B \cup C)$.

Proof: Consider arbitrary sets A, B, and C, then choose any $x \in (A \cap B) \cup C$. We will prove $x \in (A \cup C) \cap (B \cup C)$.

Since $x \in (A \cap B) \cup C$, we know that $x \in A \cap B$ or that $x \in C$. We consider each case separately.

Case 1: $\boldsymbol{x \in C}$. This in turn means that $x \in A \cup C$ and that $x \in B \cup C$.

Case 2: $\boldsymbol{x \in A \cap B}$. From $x \in A \cap B$, we learn that $x \in A$ and that $x \in B$. Therefore, we know that $x \in A \cup C$ and that $x \in B \cup C$.

In either case, we learn that $x \in A \cup C$ and $x \in B \cup C$. This establishes that $x \in (A \cup C) \cap (B \cup C)$, as required. ■

Theorem: If A, B, and C are sets, then for any $x \in (A \cap B) \cup C$, we have $x \in (A \cup C) \cap (B \cup C)$.

Proof: Consider arbitrary sets A, B, and C, then choose any $x \in (A \cap B) \cup C$. We will prove $x \in (A \cup C) \cap (B \cup C)$.

Since x … at $x \in C$.
We cons

Case … and
tha

Case … at
$x \in$
$x \in A \cup C$ and that $x \in B \cup C$.

In either case, we learn that $x \in A \cup C$ and $x \in B \cup C$. This establishes that $x \in (A \cup C) \cap (B \cup C)$, as required. ■

These are ***arbitrary choices***. Rather than specifying what ***A***, ***B***, ***C***, and ***x*** are, we're signaling to the reader that they could, in principle, supply any choices of ***A***, ***B***, ***C***, and ***x*** that they'd like.

Theorem: If A, B, and C are sets, then for any $x \in (A \cap B) \cup C$, we have $x \in (A \cup C) \cap (B \cup C)$.

Proof: Consider arbitrary sets A, B, and C, then choose any $x \in (A \cap B) \cup C$. We will prove $x \in (A \cup C) \cap (B \cup C)$.

Since $x \in (A \cap B) \cup C$, we know that $x \in A \cap B$ or that $x \in C$. We consider each case separately.

***Case 1: x ∈ C*.** This in turn means that $x \in A \cup C$ and that $x \in B \cup C$.

***Case 2: x ∈ A ∩ B*.** From $x \in A \cap B$, we learn that $x \in A$ and that $x \in B$. Therefore, we know that

In eit

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If you know that x ∈ S ∪ T:

You can conclude that $x \in S$ or that $x \in T$ (or both).

If you know that x ∈ S ∩ T:

You can conclude both that $x \in S$ and that $x \in T$.

Theorem: If A, B, and C are sets, then for any $x \in (A \cap B) \cup C$,

To prove that x ∈ S ∪ T:
Prove either that $x \in S$ or that $x \in T$ (or both).

To prove that x ∈ S ∩ T:
Prove both that $x \in S$ and that $x \in T$.

We consider each case separately.

Case 1: ***x ∈ C***. This in turn means that $x \in A \cup C$ and that $x \in B \cup C$.

Case 2: ***x ∈ A ∩ B***. From $x \in A \cap B$, we learn that $x \in A$ and that $x \in B$. Therefore, we know that $x \in A \cup C$ and that $x \in B \cup C$.

In either case, we learn that $x \in A \cup C$ and $x \in B \cup C$. This establishes that $x \in (A \cup C) \cap (B \cup C)$, as required. ■

Theorem: If A, B, and C are sets, then for any $x \in (A \cap B) \cup C$, we have $x \in (A \cup C) \cap (B \cup C)$.

Proof: Consider arbitrary sets A, B, and C, then choose any $x \in (A \cap B) \cup C$. We will prove $x \in (A \cup C) \cap (B \cup C)$.

Since $x \in (A \cap B) \cup C$, we know that $x \in A \cap B$ or that $x \in C$. We consider each case separately.

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$x \in A \cup C$ and th

In either case, we lear

establishes that $x \in (A$

This is called a **proof by cases** (alternatively, a **proof by exhaustion**) and works by showing that the theorem is true regardless of what specific outcome arises.

Theorem: If A, B, and C are sets, then for any $x \in (A \cap B) \cup C$, we have $x \in (A \cup C) \cap (B \cup C)$.

Proof: Consider arbitrary
y
$x \in (A \cap B) \cup C$. We w

Since $x \in (A \cap B) \cup C$,
$\in C$.
We consider each case

After splitting into cases, it's a good idea to summarize what you just did so that the reader knows what to take away from it.

Case 1: $x \in C$. This in turn means that $x \in A \cup C$ and that $x \in B \cup C$.

Case 2: $x \in A \cap B$. From $x \in A \cap B$, we learn that $x \in A$ and that $x \in B$. Therefore, we know that $x \in A \cup C$ and that $x \in B \cup C$.

In either case, we learn that $x \in A \cup C$ and $x \in B \cup C$. This establishes that $x \in (A \cup C) \cap (B \cup C)$, as required. ■

Theorem: If A, B, and C are sets, then for any $x \in (A \cap B) \cup C$, we have $x \in (A \cup C) \cap (B \cup C)$.

Proof: Consider arbitrary sets A, B, and C, then choose any $x \in (A \cap B) \cup C$. We will prove $x \in (A \cup C) \cap (B \cup C)$.

Since $x \in (A \cap B) \cup C$, we know that $x \in A \cap B$ or that $x \in C$. We consider each case separately.

Case 1: $x \in C$. This in turn means that $x \in A \cup C$ and that $x \in B \cup C$.

Case 2: $x \in A \cap B$. From $x \in A \cap B$, we learn that $x \in A$ and that $x \in B$. Therefore, we know that $x \in A \cup C$ and that $x \in B \cup C$.

In either case, we learn that $x \in A \cup C$ and $x \in B \cup C$. This establishes that $x \in (A \cup C) \cap (B \cup C)$, as required. ■

Proofs as a Dialog

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Let n be an arbitrary odd integer.

Since n is an odd integer, there is an integer k such that $n = 2k + 1$.

Now, let $z = k - 34$.

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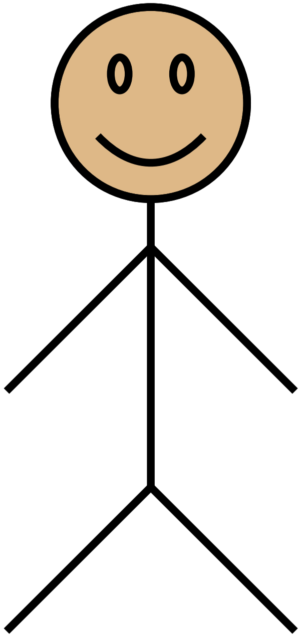

Proof Writer (You)

Proofs as a Dialog

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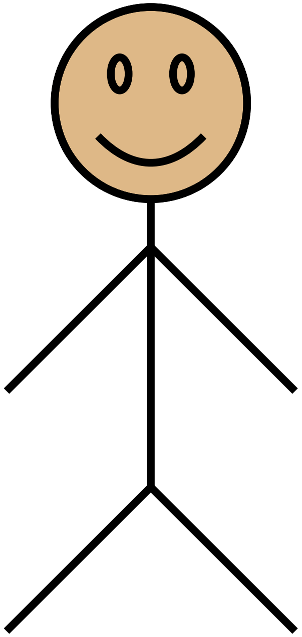
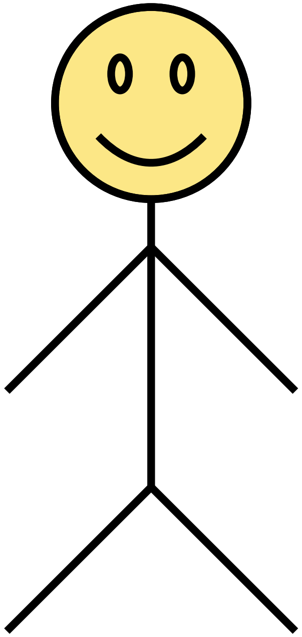


Proof Writer (You)



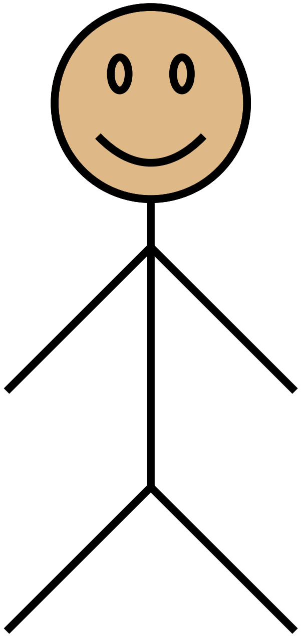
Proof Reader

Proofs as a Dialog

Let n be an arbitrary odd integer.

Since n is an odd integer, there is an integer k such that $n = 2k + 1$.

Now, let $z = k - 34$.



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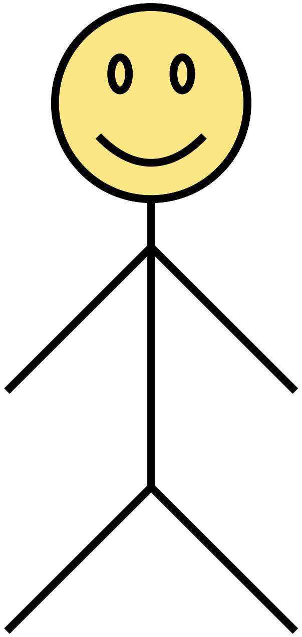

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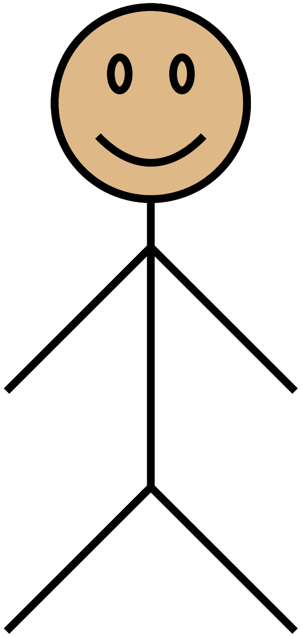
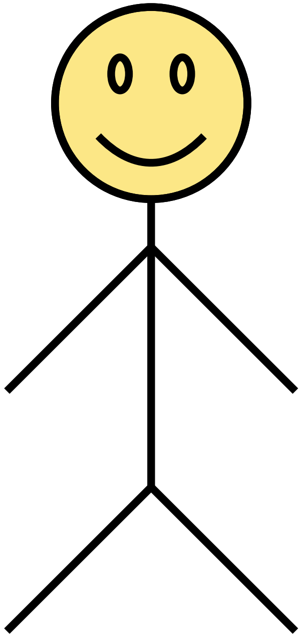


Proof Writer (You)

$n = 137$

Reader Picks



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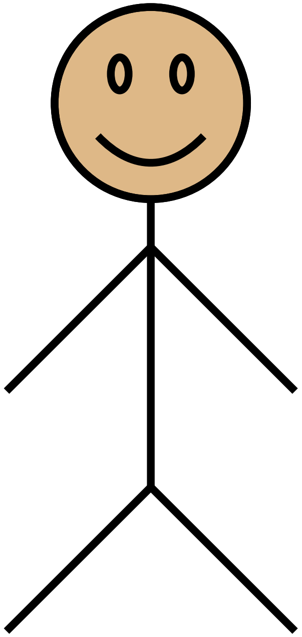
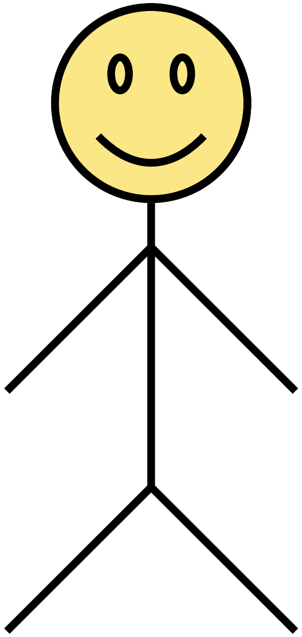


Proof Writer (You)

$n = 137$

Reader *Picks*



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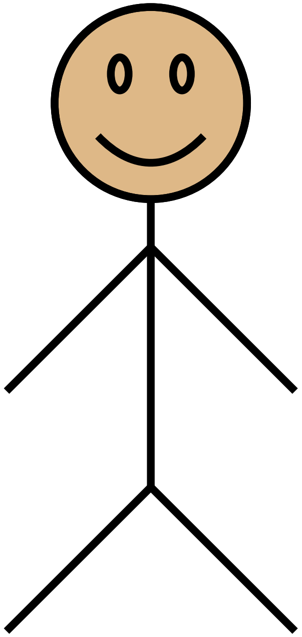


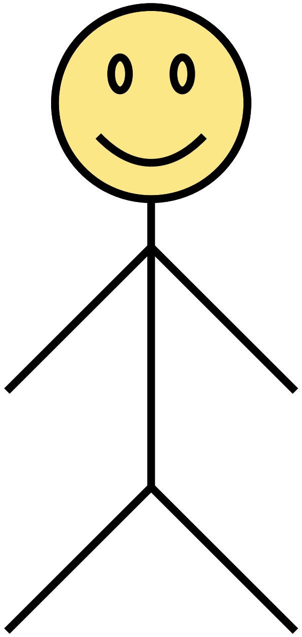
Proof Writer (You)

$n = 137$

Reader *Picks*

$k = 68$

Neither *Picks*



Proof Reader

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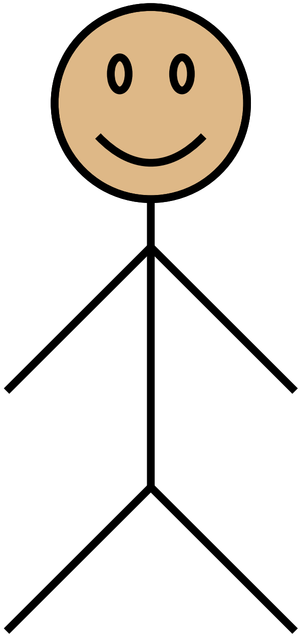

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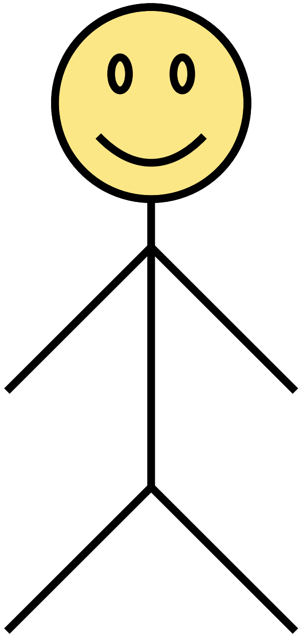

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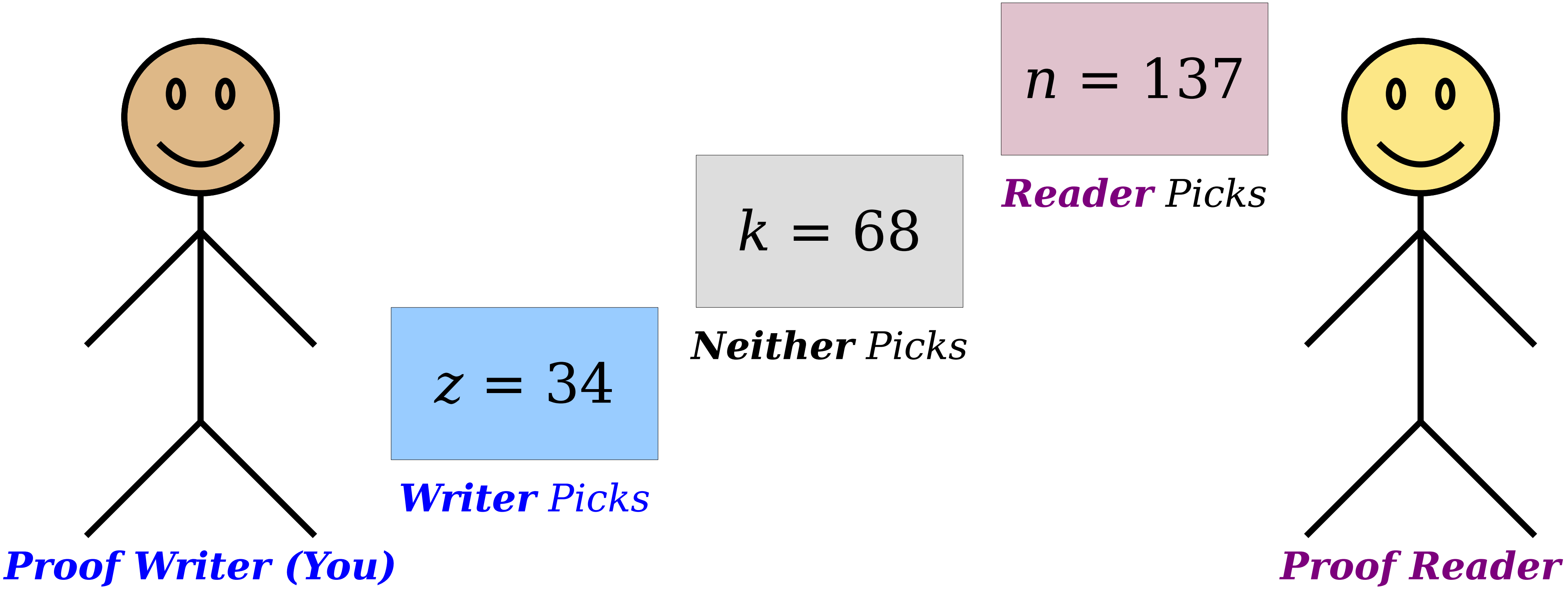

$n = 137$

Reader *Picks*

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Neither *Picks*

$z = 34$

Writer *Picks*

Proof Writer (You)

Proof Reader

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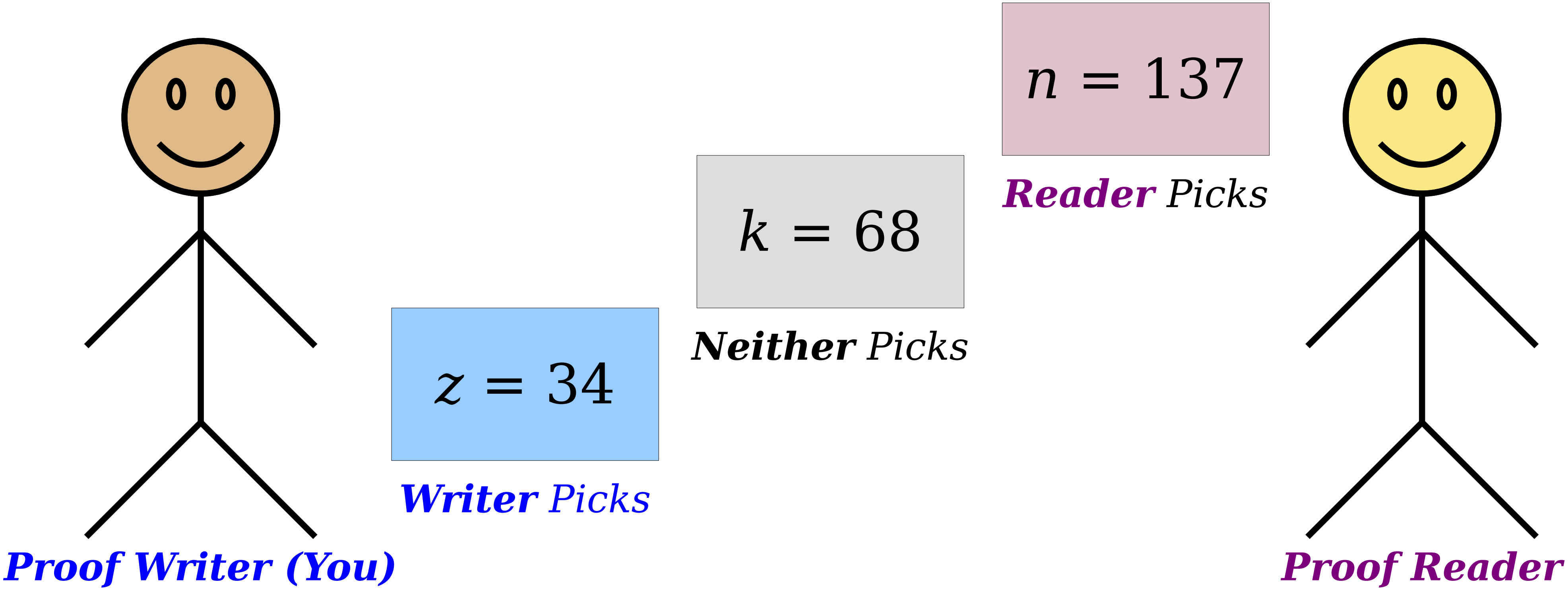

$z = 34$

***Writer** Picks*

$k = 68$

***Neither** Picks*

$n = 137$

***Reader** Picks*

Proof Writer (You)

Proof Reader

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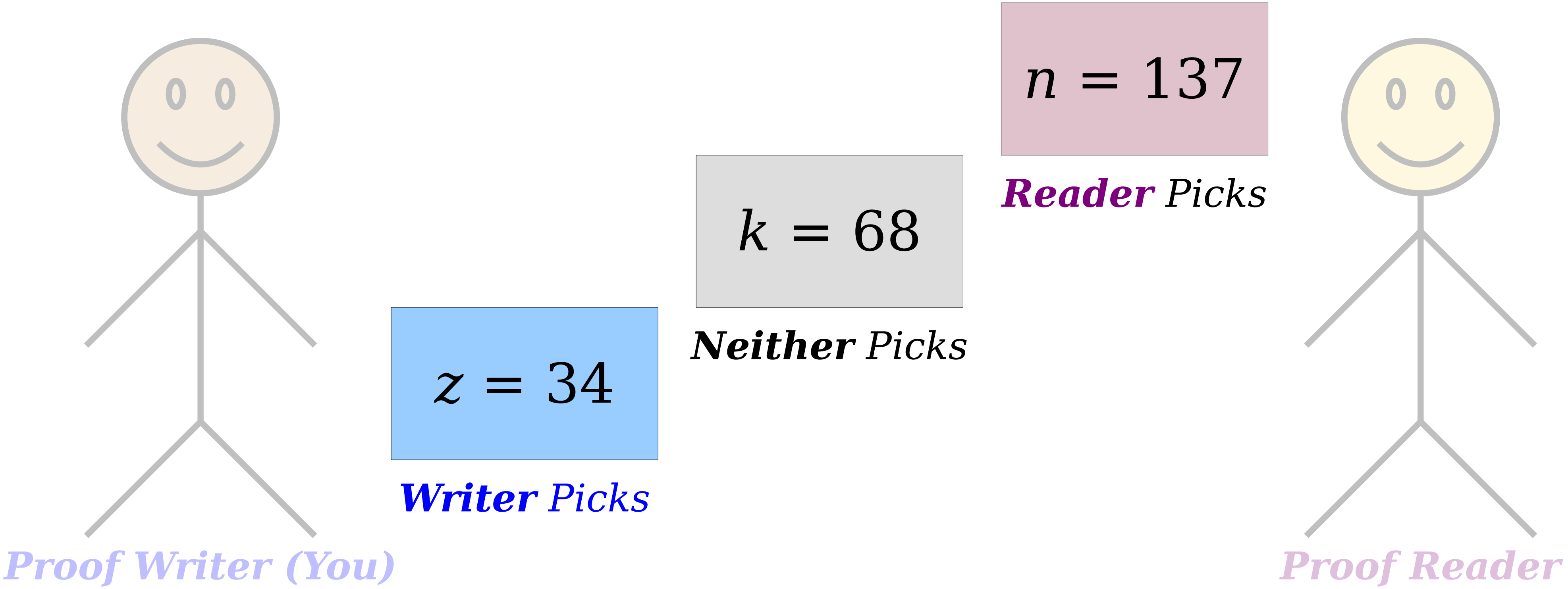


$n = 137$

Reader *Picks*

$k = 68$

Neither *Picks*

$z = 34$

Writer *Picks*

Proof Writer (You)

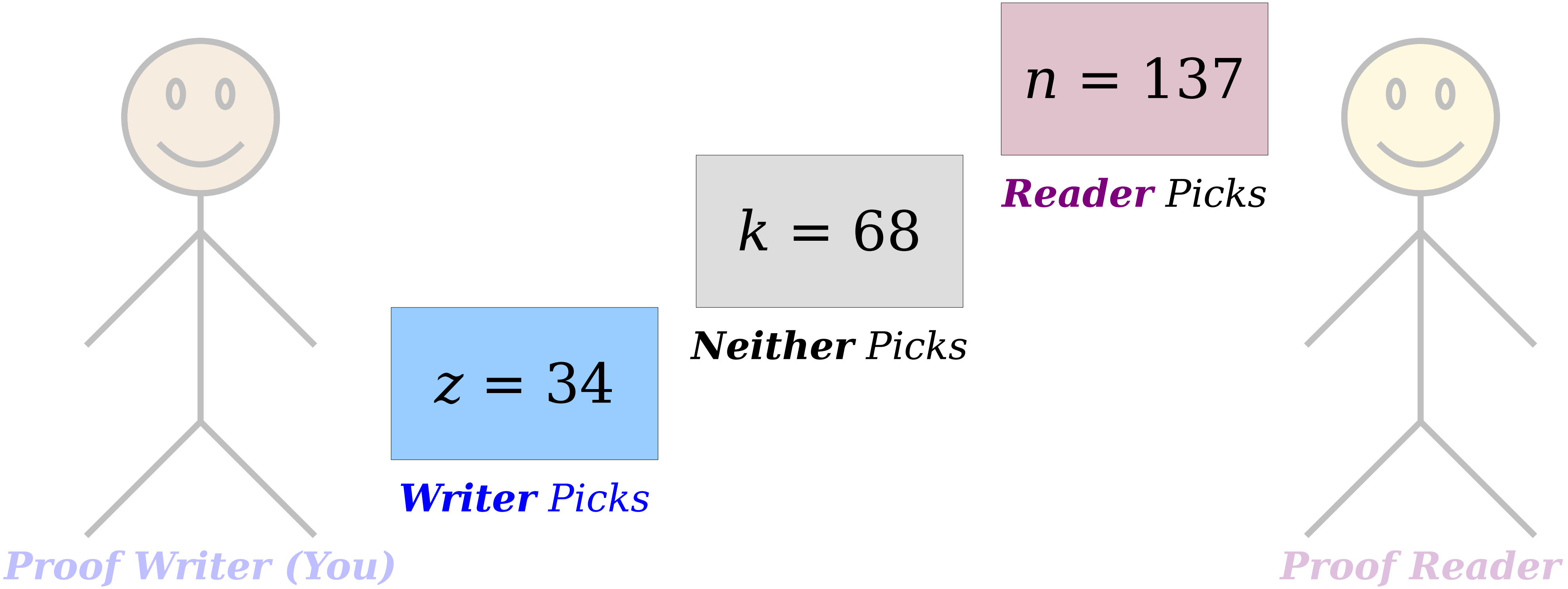
Proof Reader

Each of these variables has a distinct, assigned value.

Each variable was either picked by the reader, picked by the writer, or has a value that can be determined from other variables.

Sinc r

Now, let $z = k - 34$.



$n = 137$

Reader *Picks*

$k = 68$

Neither *Picks*

$z = 34$

Writer *Picks*

Proof Writer (You)

Proof Reader

Who Owns What?

- The ***reader*** chooses and owns a value if you use wording like this:
  - Pick a natural number n.
  - Consider some $n \in \mathbb{N}$.
  - Fix a natural number n.
  - Let n be a natural number.
- The ***writer*** (you) chooses and owns a value if you use wording like this:
  - Let $r = n + 1$.
  - Pick $s = n$.
- ***Neither*** of you chooses a value if you use wording like this:
  - Since n is even, we know there is some $k \in \mathbb{Z}$ where $n = 2k$.
  - Because n is odd, there must be some integer k where $n = 2k + 1$.

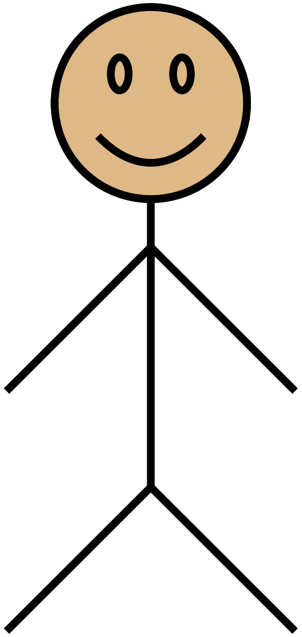
Proofs as a Dialog

⚠

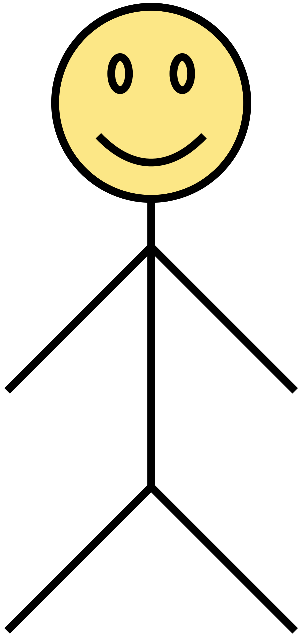
Let x be an arbitrary even integer.

Then for any even x, we know that $x+1$ is odd.

⚠



Proof Writer (You)



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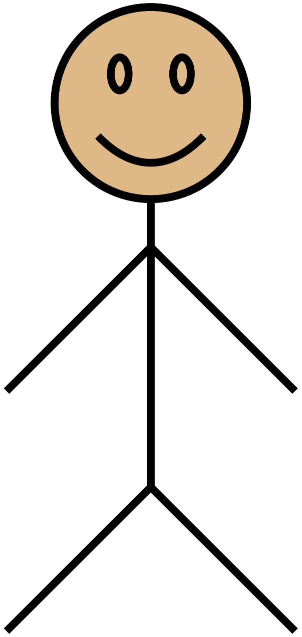

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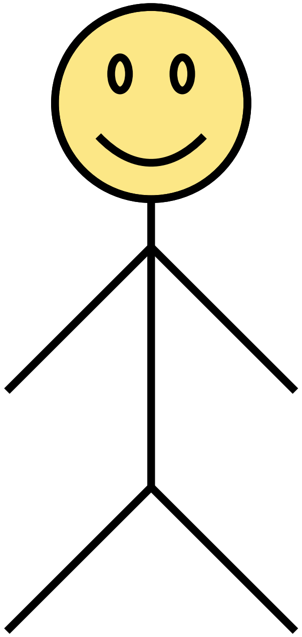

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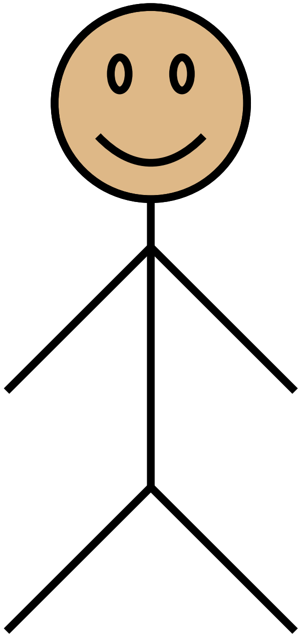
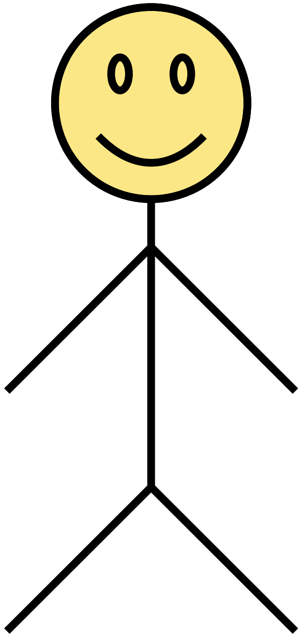


Proof Writer (You)

$x = 242$

Reader *Picks*



Proof Reader

Proofs as a Dialog

⚠ ⚠

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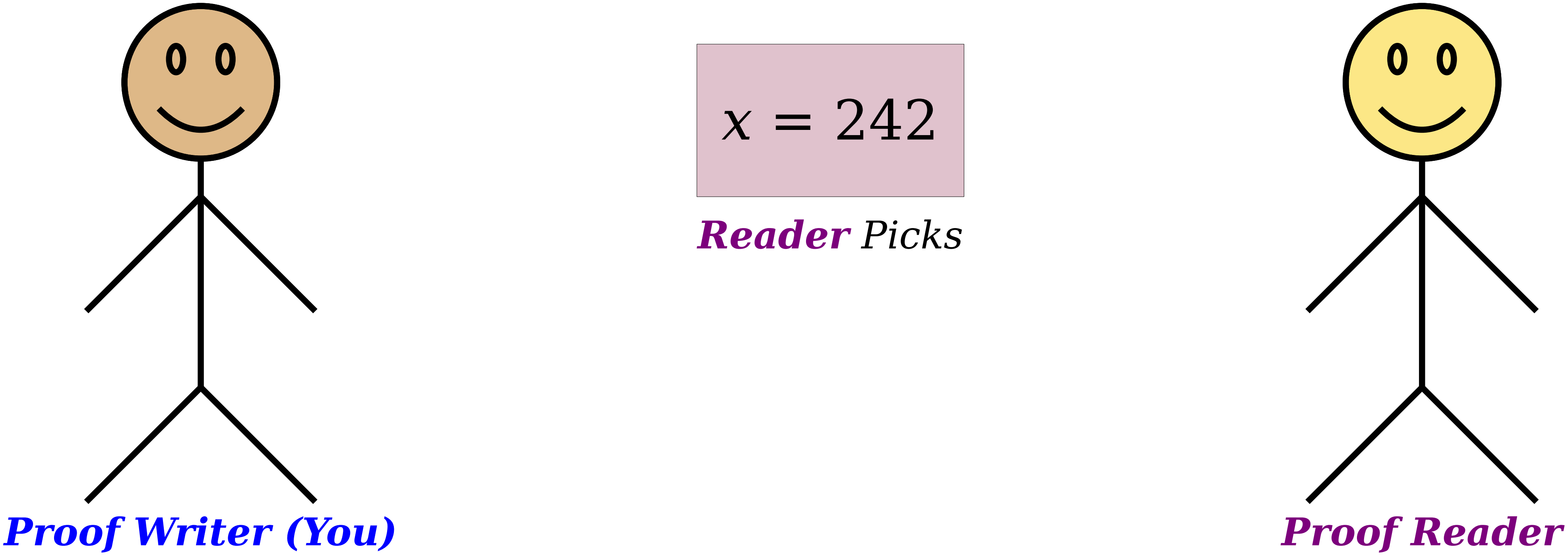

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Proof Reader

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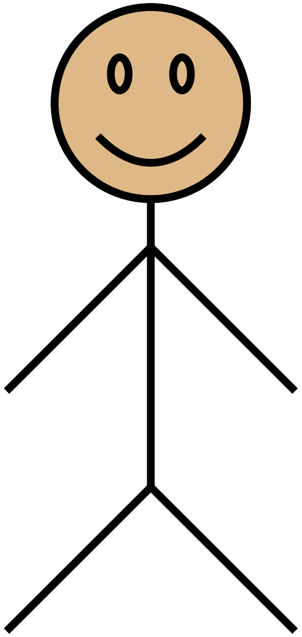

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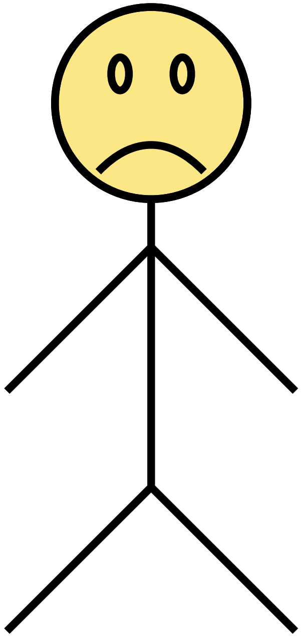

Proof Reader

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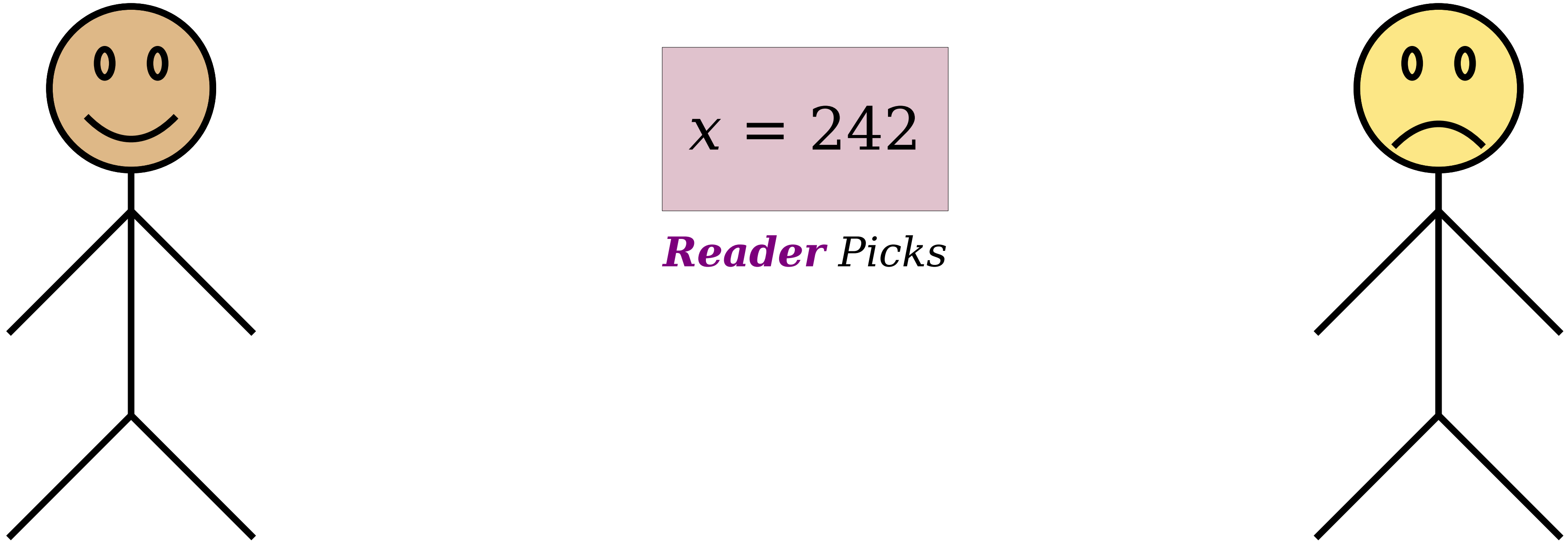


$x = 242$

Reader *Picks*

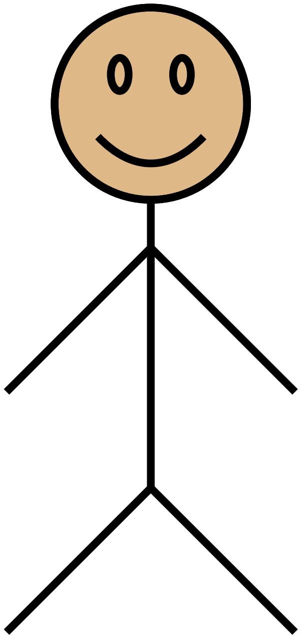
Proof Writer (You)

Proof Reader

Proofs as a Dialog

⚠ Let x be an arbitrary even integer. ⚠

Then **for any even x**, we know that $x+1$ is odd.



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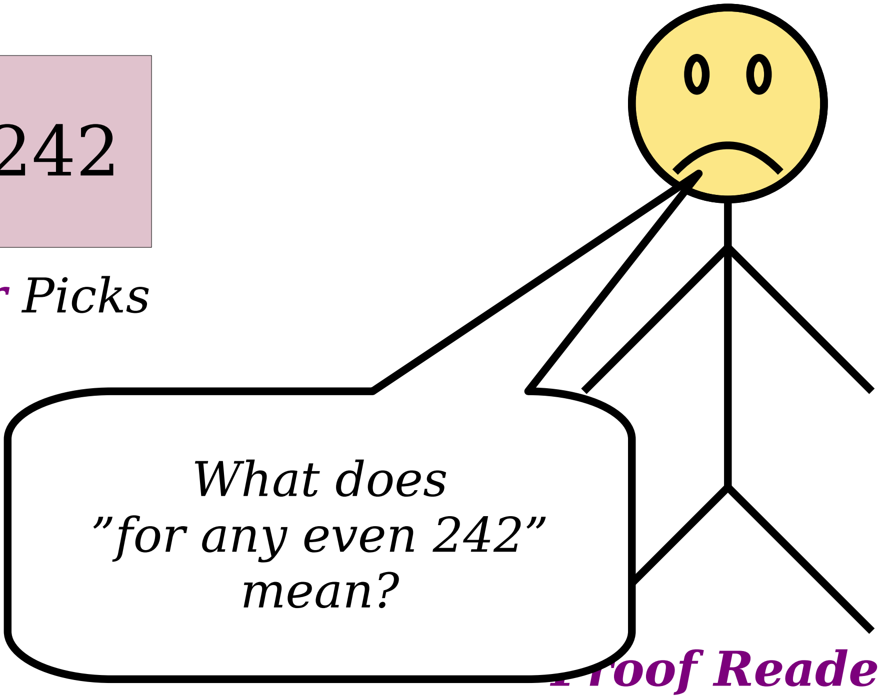
Reader *Picks*



Proof Reader

Proofs as a Dialog

Let x be an arbitrary even integer.

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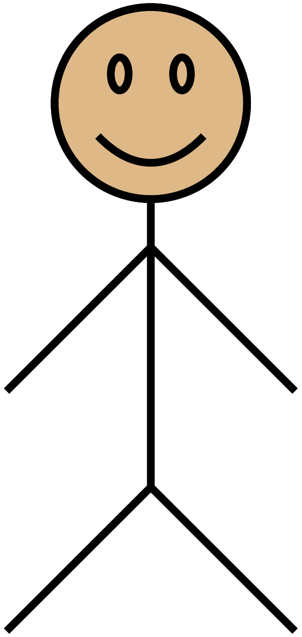

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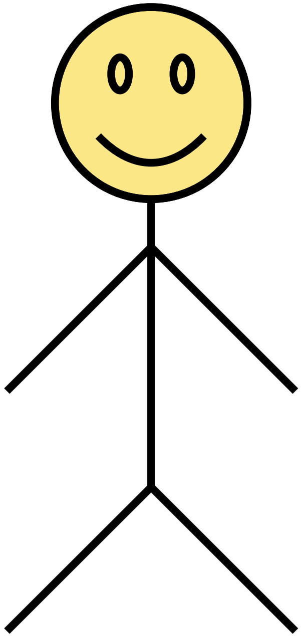

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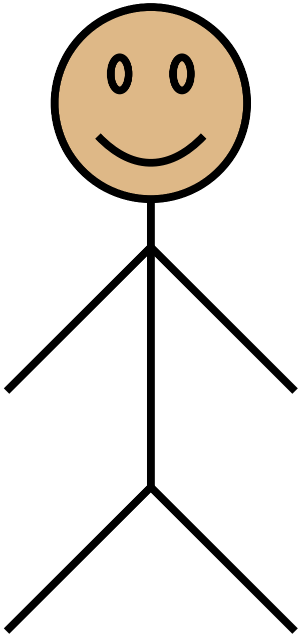

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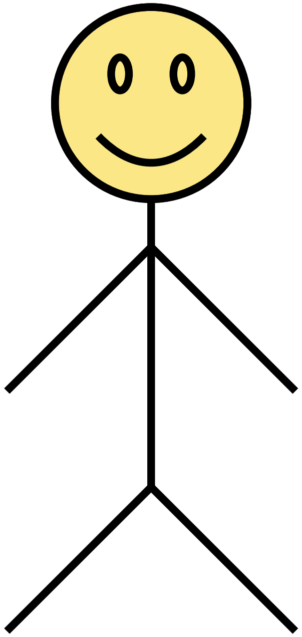

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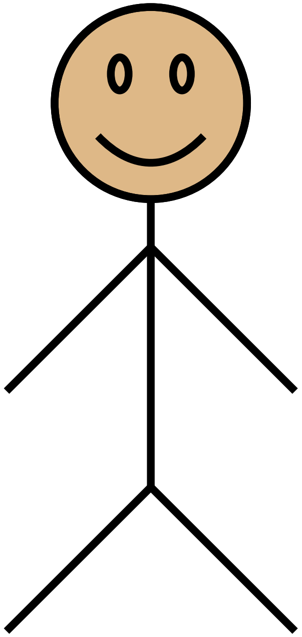

Proof Writer (You)

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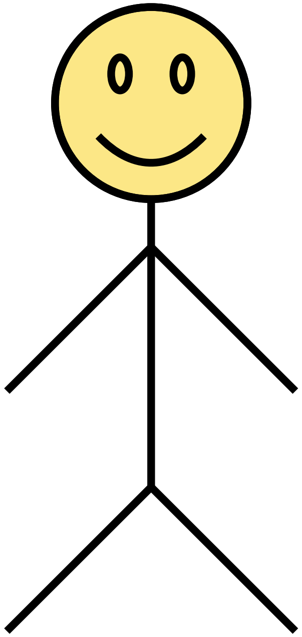
***Reader** Picks*



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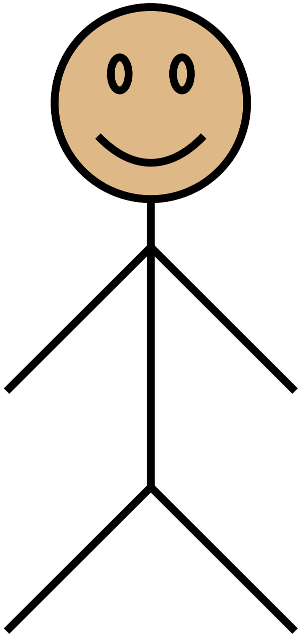

Proof Writer (You)

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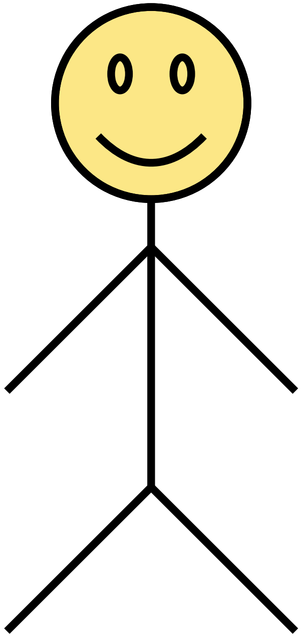


Proof Reader

Every variable needs a value.

Avoid talking about “all x” or “every x” when manipulating something concrete.

To prove something is true for any choice of a value for x, let the reader pick x.

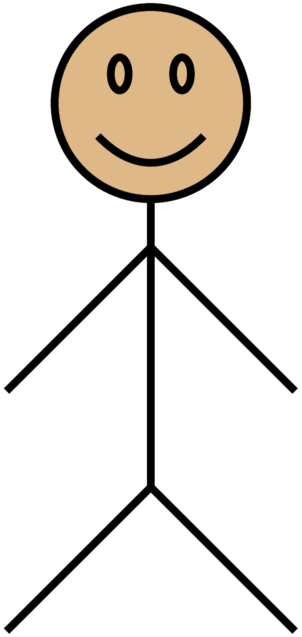
Once you've said something like

Let x be an integer.
Consider an arbitrary $x \in \mathbb{Z}$.
Pick any x.

Do not say things like the following:

This means that ***for any*** $x \in \mathbb{Z}$ …
So ***for all*** $x \in \mathbb{Z}$ …

Proofs as a Dialog



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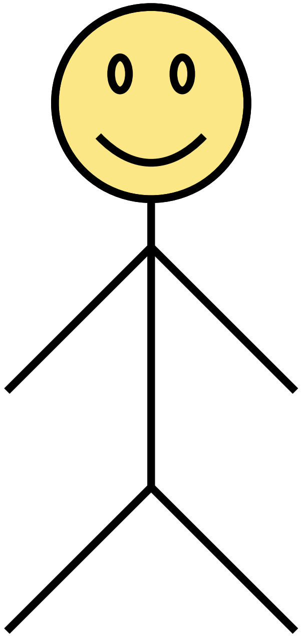


Proof Reader

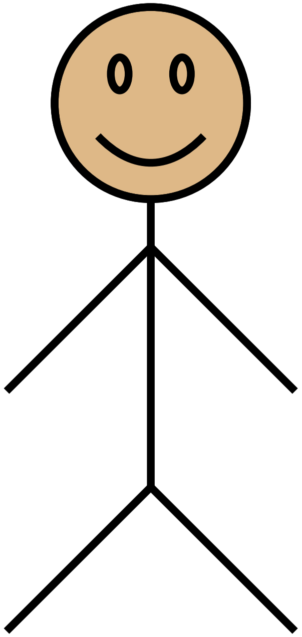
Proofs as a Dialog

⚠

Pick two integers m and n where $m+n$ is odd.

Let $n = 1$, which means that $m+1$ is odd.

⚠



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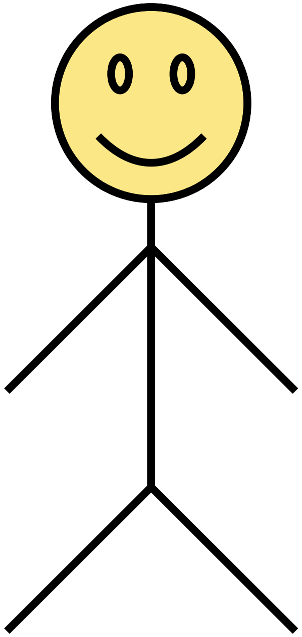

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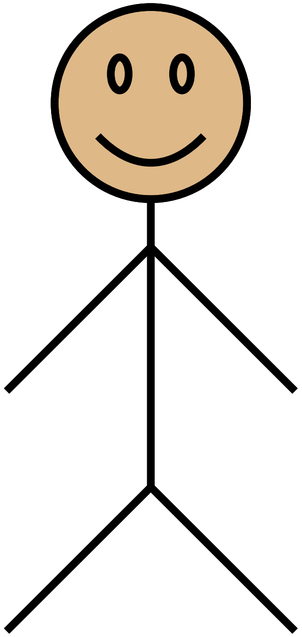

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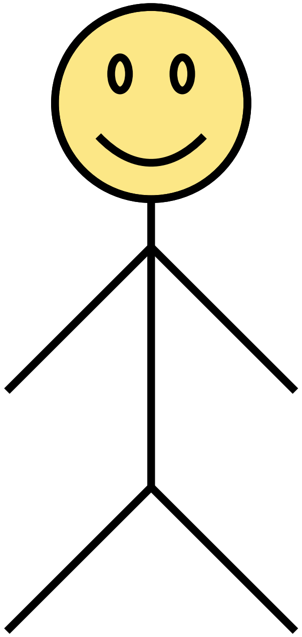

Proof Reader

Proofs as a Dialog

⚠ ⚠

Pick two integers m and n where $m+n$ is odd.

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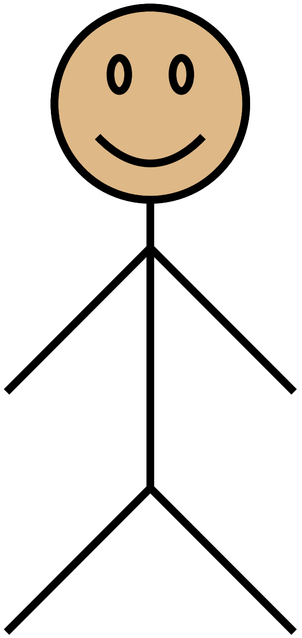
$m = 103$

Reader *Picks*

$n = 166$

Reader *Picks*



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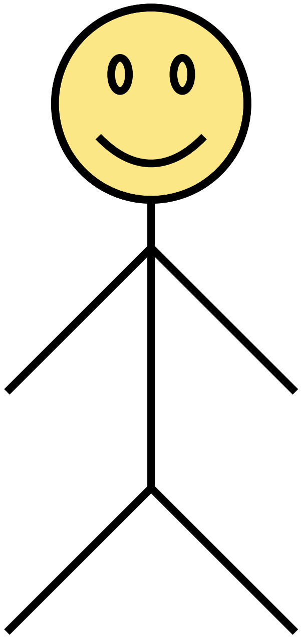

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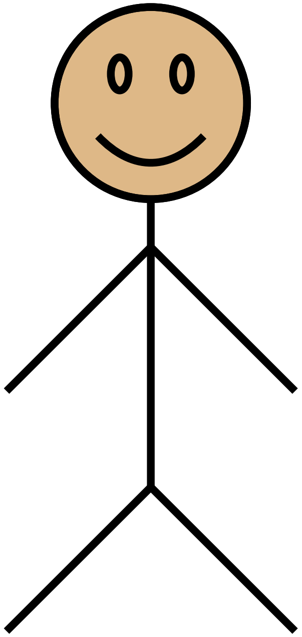


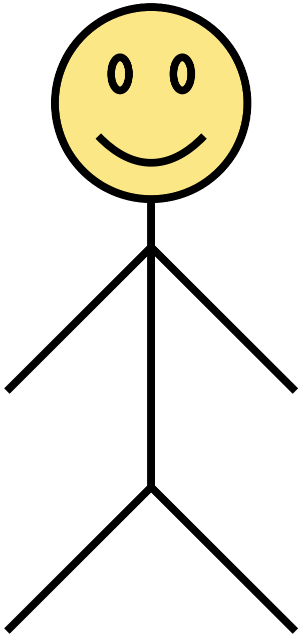
Proof Writer (You)

$m = 103$

***Reader** Picks*

$n = 166$

***Reader** Picks*



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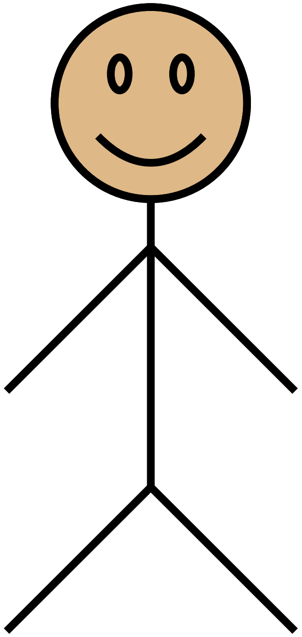

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Reader *Picks*

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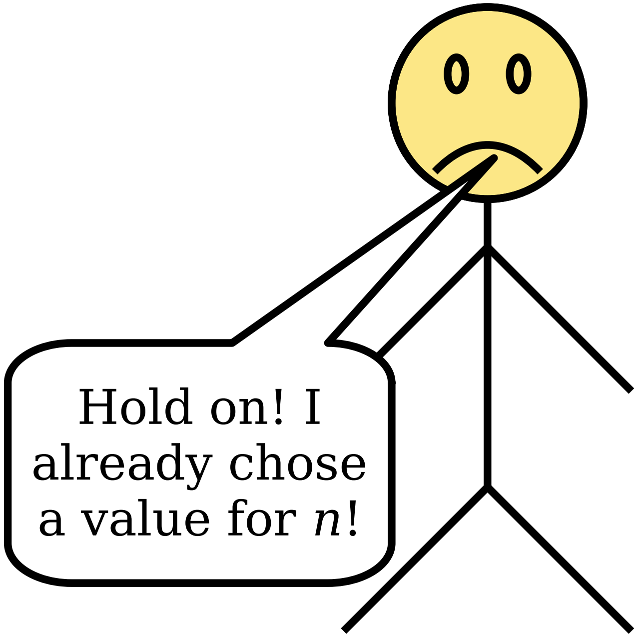
Reader *Picks*



Proof Reader

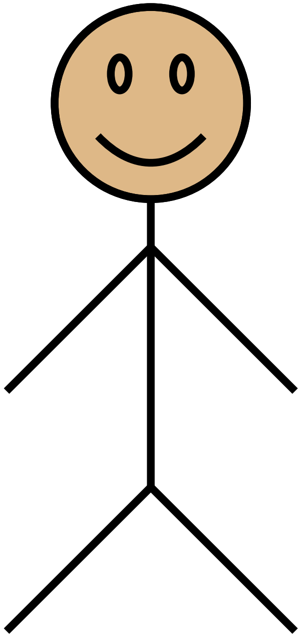
Proofs as a Dialog

⚠

Let $n = 1$.

Pick any integer m where $m+1$ is odd.

⚠



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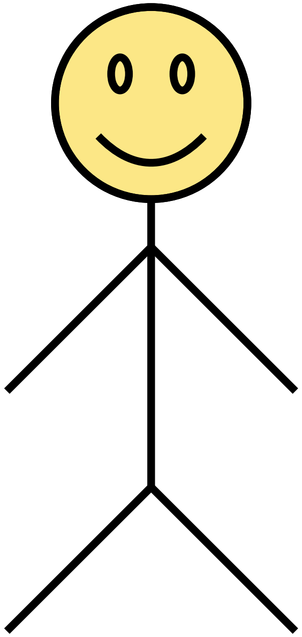

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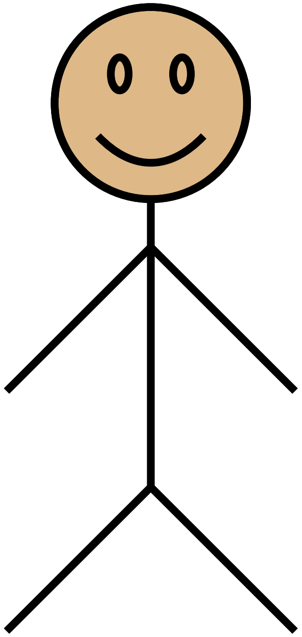

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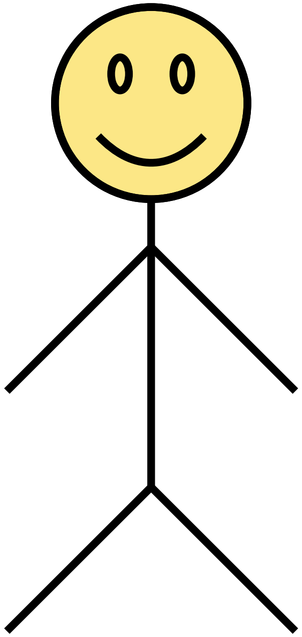

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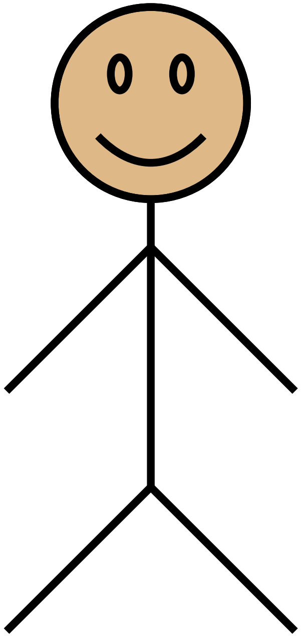
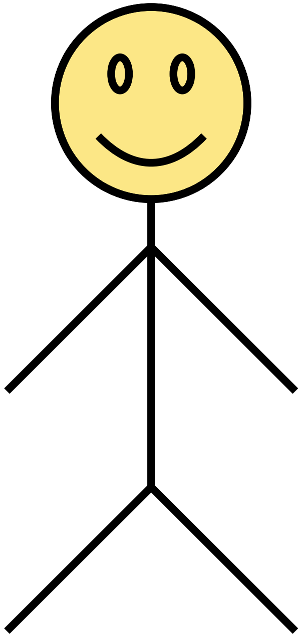


Proof Writer (You)

$n = 1$

Writer *Picks*



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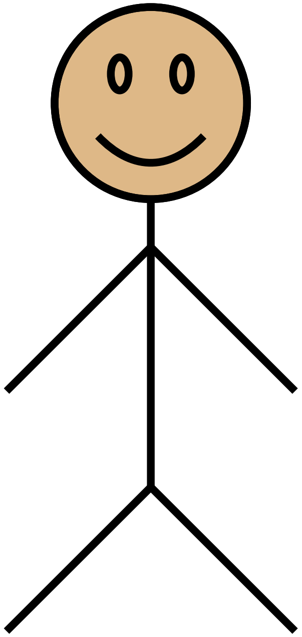
⚠

Let $n = 1$.

Pick any integer m where $m+1$ is odd.

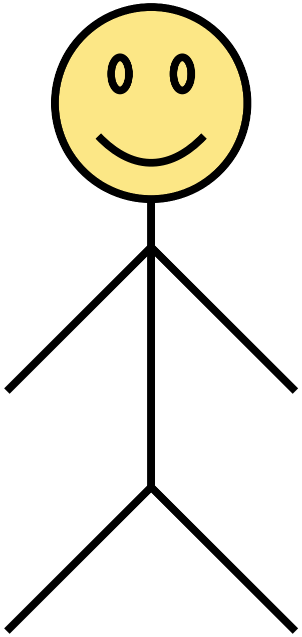
⚠



Proof Writer (You)

$n = 1$

Writer *Picks*



Proof Reader

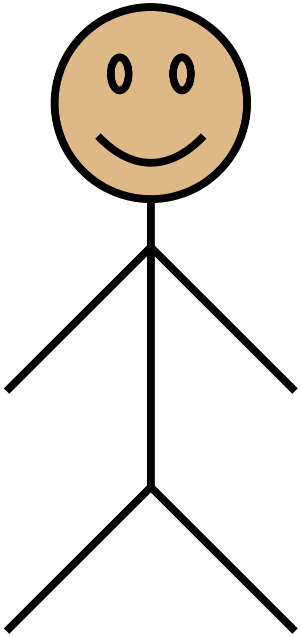
Proofs as a Dialog

⚠

Let $n = 1$.

Pick any integer m where $m+1$ is odd.

⚠



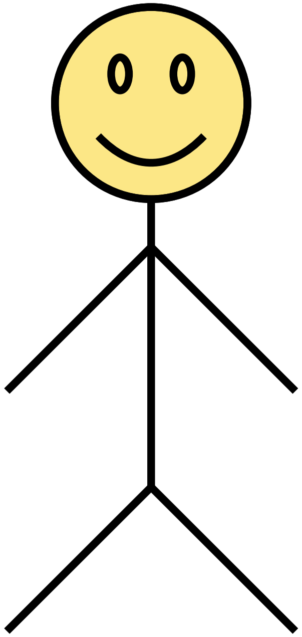
Proof Writer (You)

$m = 166$

Reader *Picks*

$n = 1$

Writer *Picks*



Proof Reader

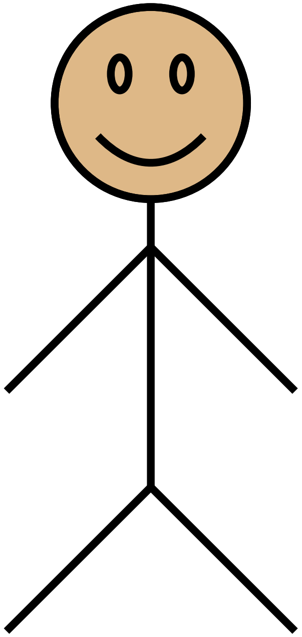
Proofs as a Dialog

⚠

Let $n = 1$.

Pick any integer m where $m+1$ is odd.

⚠



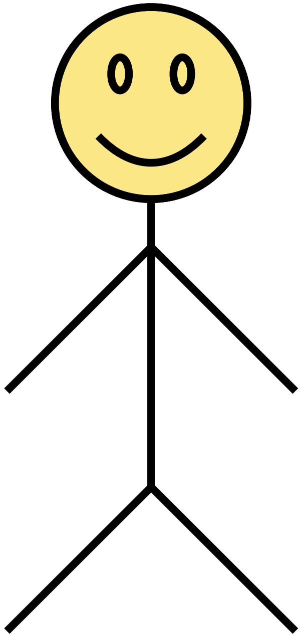
Proof Writer (You)

$m = 166$

Reader *Picks*

$n = 1$

Writer *Picks*



Proof Reader

Proofs as a Dialog

⚠

Let $n = 1$.

Pick any integer m where $m+1$ is odd.

Do we even need n here?

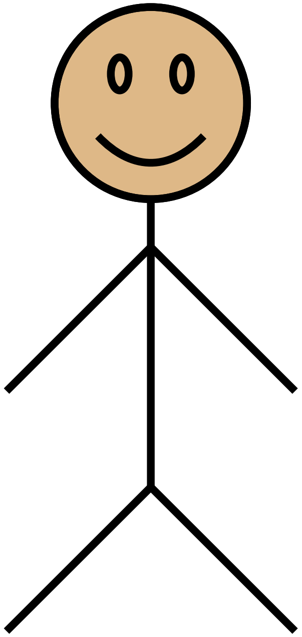
⚠

$m = 166$

Reader *Picks*

$n = 1$

Writer *Picks*



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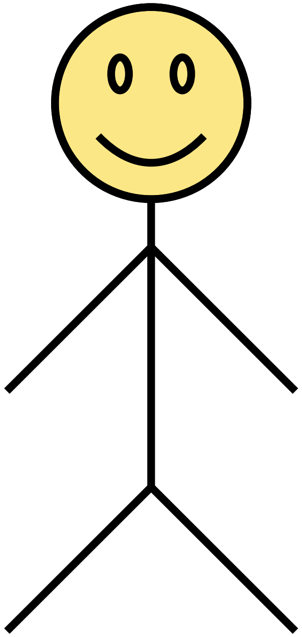
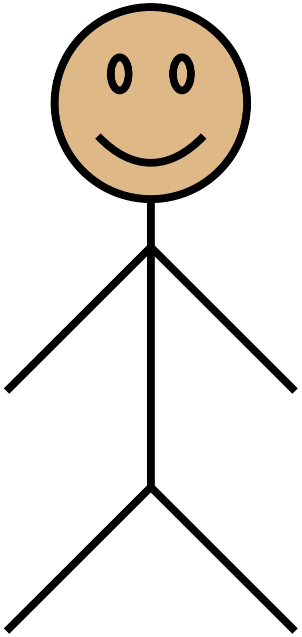


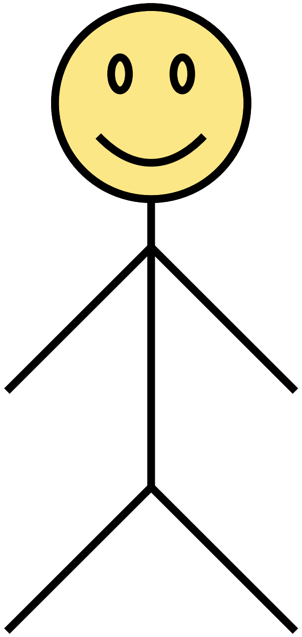
Proof Reader

Proofs as a Dialog

Pick any integer m where $m+1$ is odd.



Proof Writer (You)



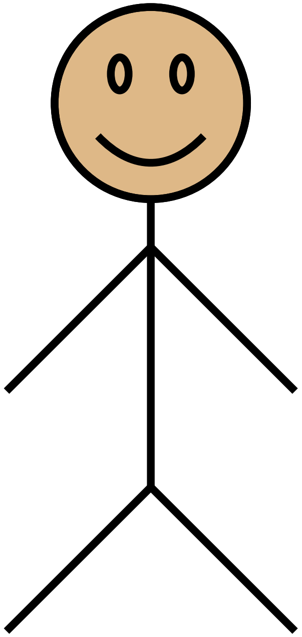
Proof Reader

Proofs as a Dialog

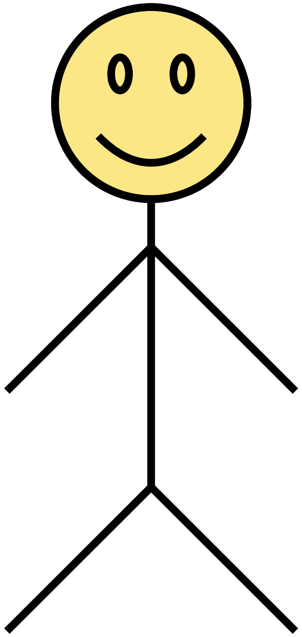
Pick any integer m where $m+1$ is odd.

$m = 166$

***Reader** Picks*



Proof Writer (You)



Proof Reader

Be mindful of who owns what variable.

Don’t change something you don’t own.

You don’t always need to name things, especially if they already have a name.

Your Action Items

- ***Read "How to Succeed in CS103."***
  - There's a lot of valuable advice in there – take it to heart!
- ***Read "Guide to ∈ and ⊆."***
  - You'll want to have a handle on how these concepts are related, and on how they differ.
- ***Finish and submit Problem Set 0.***
  - Don't put this off until the last minute!

Next Time

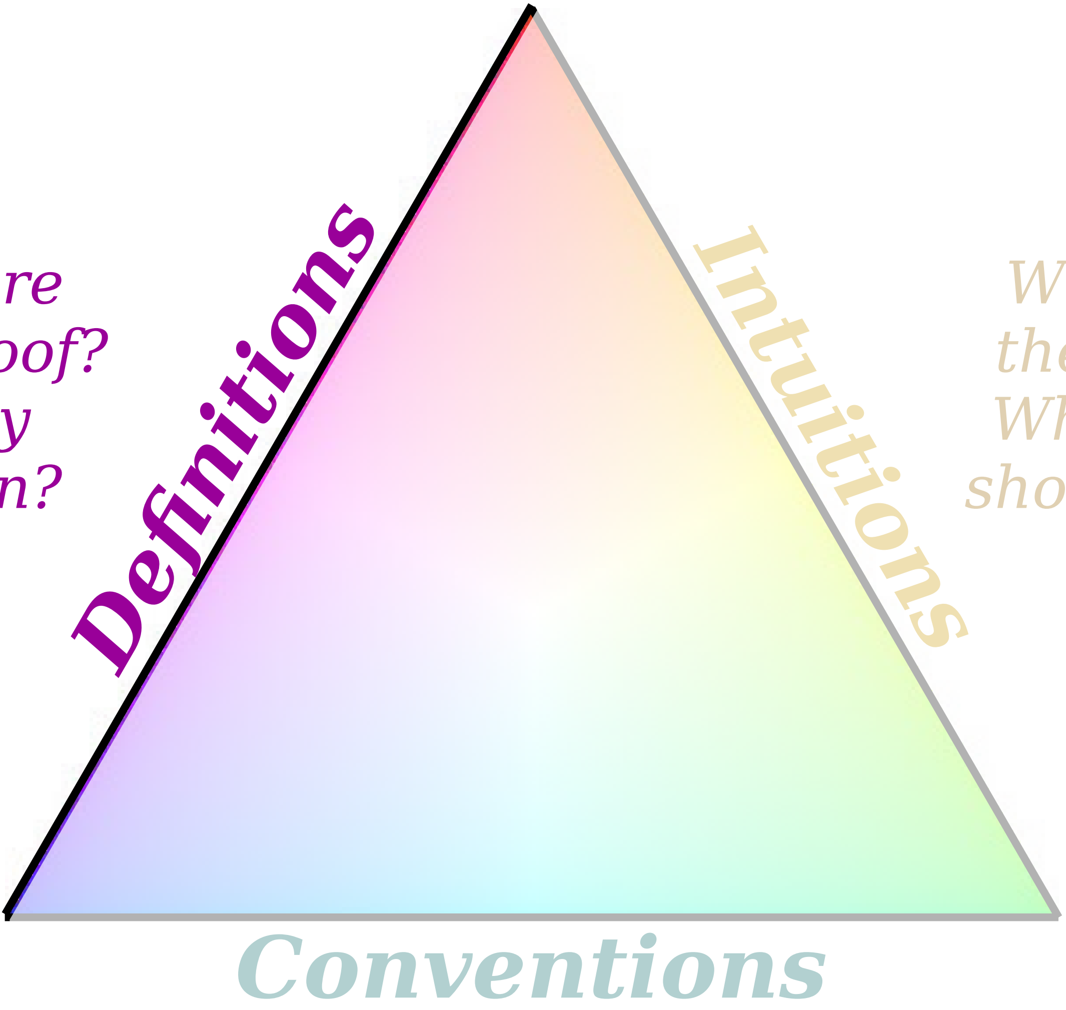
- ***Indirect Proofs***
  - How do you prove something without actually proving it?
- ***Mathematical Implications***
  - What exactly does “if *P*, then *Q*” mean?
- ***Proof by Contrapositive***
  - A helpful technique for proving implications.
- ***Proof by Contradiction***
  - Proving something is true by showing it can't be false.

Appendix: More Proofs on Sets

Proofs on Subsets

Theorem: If A, B, and C are sets, then $(A \cup C) \cap (B \cup C) \subseteq (A \cap B) \cup C$.

What terms are used in this proof? What do they formally mean?



What does this theorem mean? Why, intuitively, should it be true?

What is the standard format for writing a proof? What are the techniques for doing so?

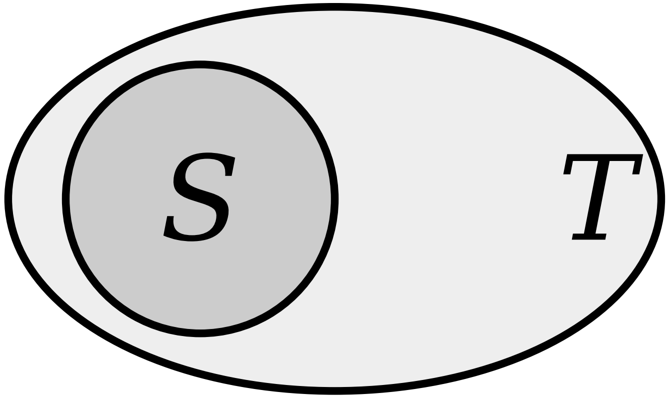
Set Theory Review

- Recall from last time that we write $x \in S$ if x is an element of set S and $x \notin S$ if x is not an element of set S.
- If S and T are sets, we say that S is a subset of T (denoted $S \subseteq T$) if the following statement is true:

For every x, if $x \in S$, then $x \in T$.

- What does this mean for proofs?

Subsets

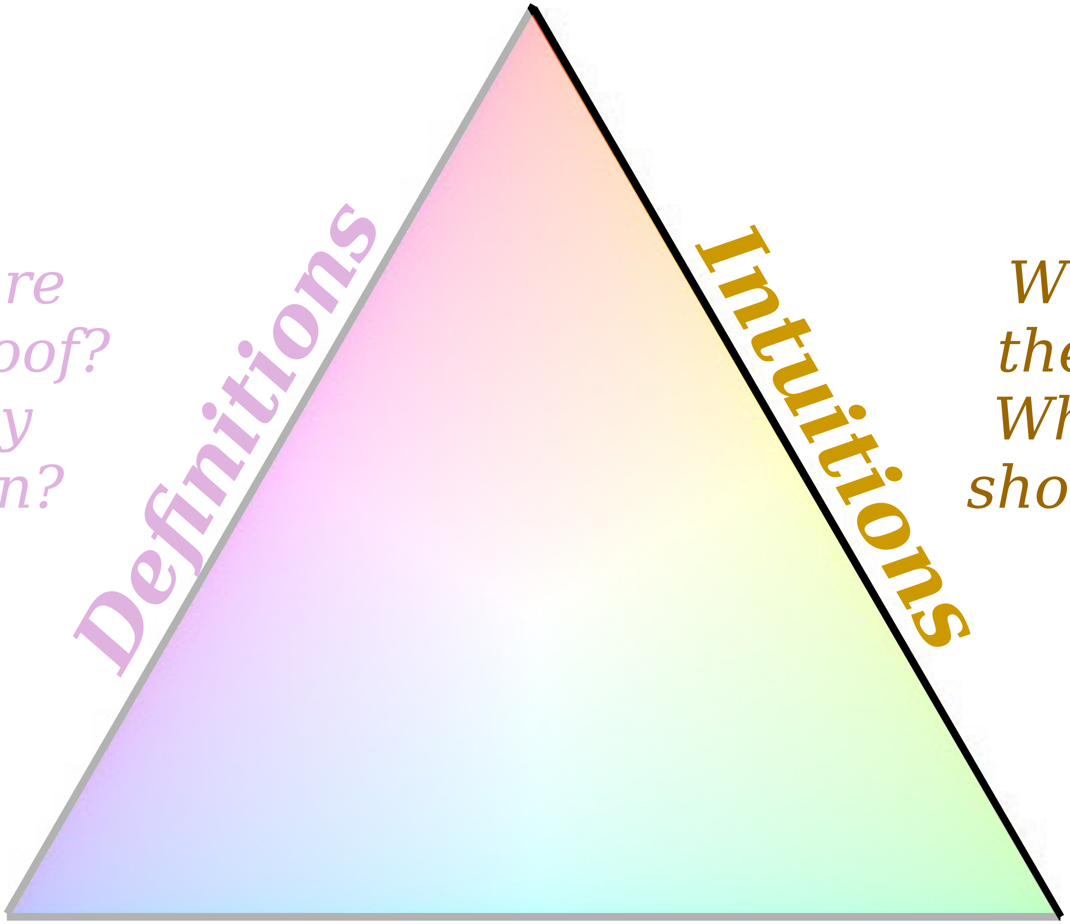


$S \subseteq T$

Definition: If S and T are sets, then $S \subseteq T$ when for every $x \in S$, we have $x \in T$.

To prove that $S \subseteq T$:
Pick an arbitrary $x \in S$, then prove $x \in T$.

If you know that $S \subseteq T$:
If you have an $x \in S$, you can conclude $x \in T$.



What terms are
used in this proof?
What do they
formally mean?

Definitions

Intuitions

What does this
theorem mean?
Why, intuitively,
should it be true?

Conventions

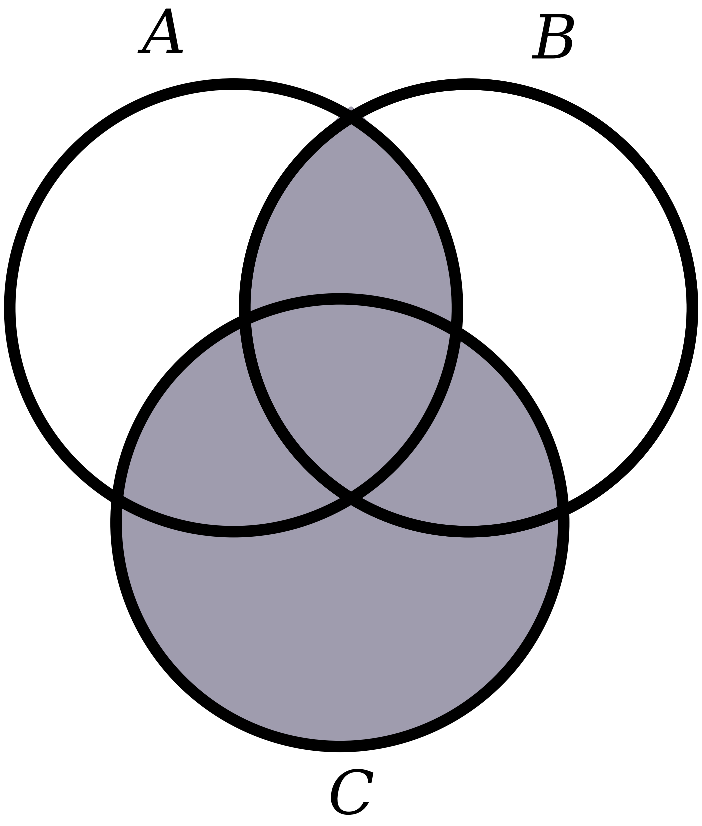
What is the standard
format for writing a proof?
What are the techniques
for doing so?

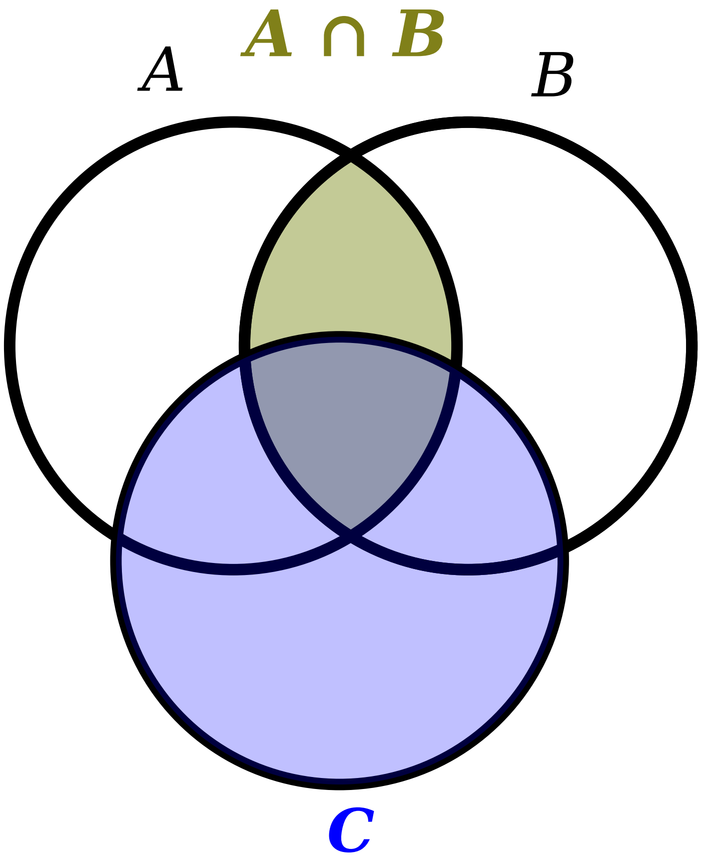
Theorem: If A, B, and C are sets, then $(A \cup C) \cap (B \cup C) \subseteq (A \cap B) \cup C$.

Let’s Draw Some Pictures!

Theorem: If A, B, and C are sets,
then $(A \cup C) \cap (B \cup C) \subseteq (A \cap B) \cup C$.

Let’s Draw Some Pictures!





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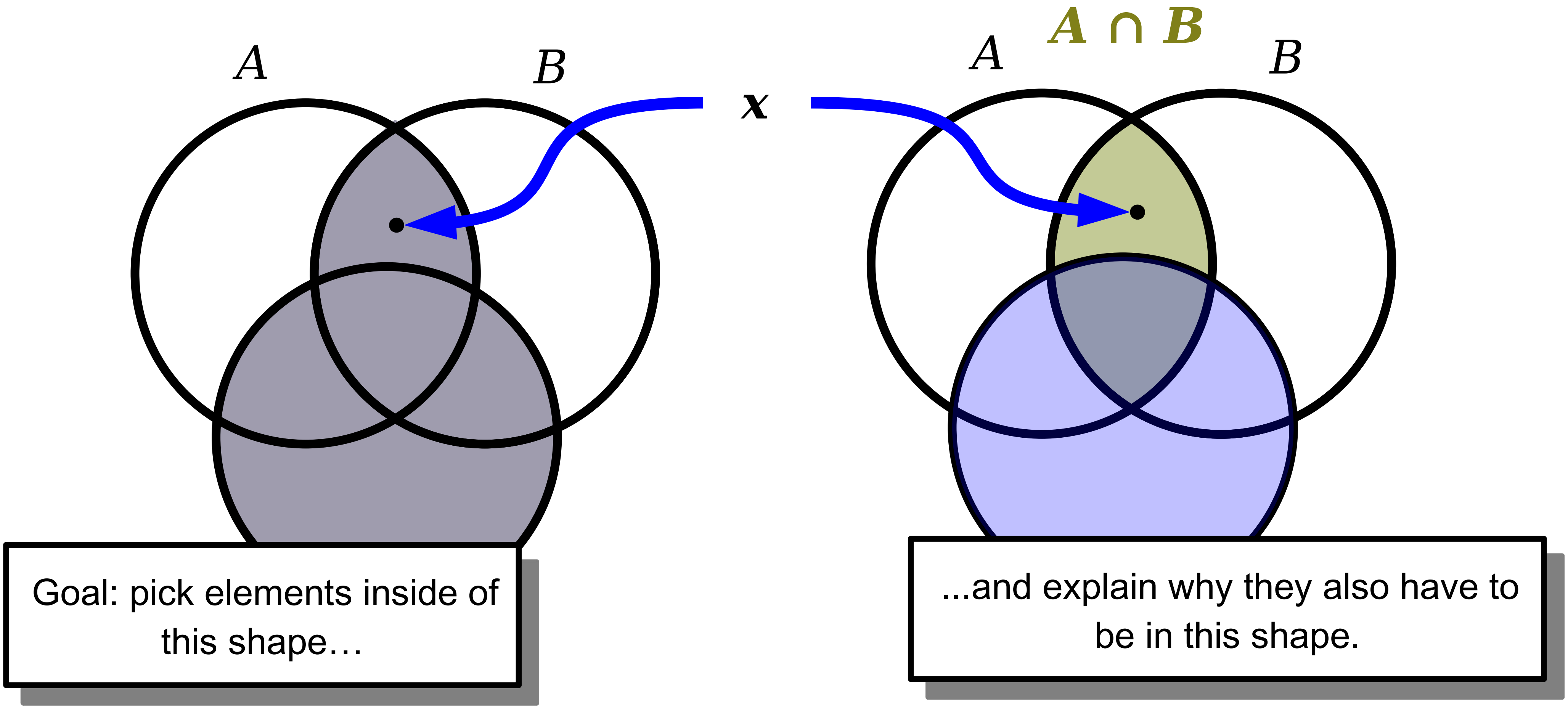
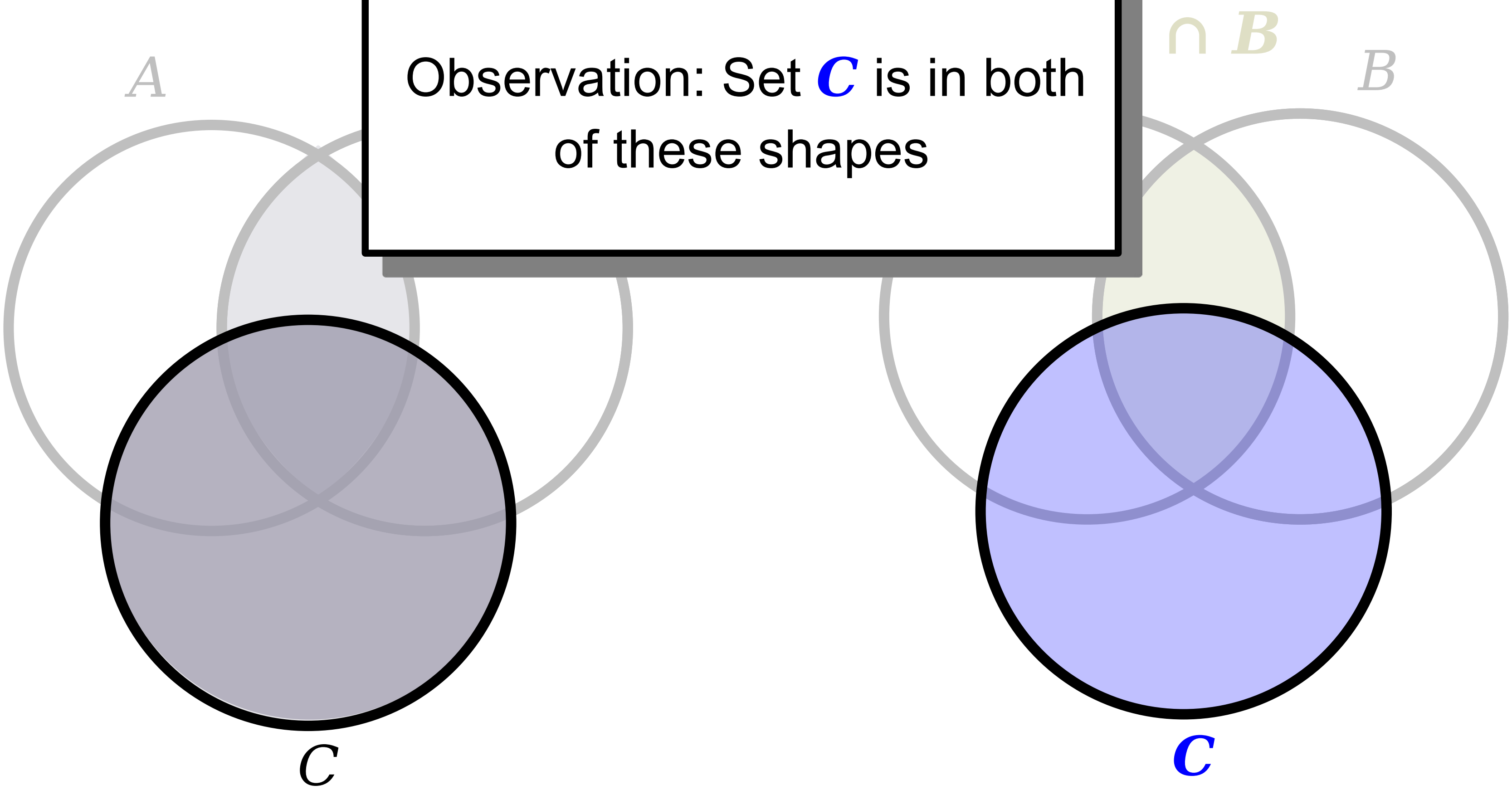


Goal: pick elements inside of this shape…

...and explain why they also have to be in this shape.

Theorem: If A, B, and C are sets, then $(A \cup C) \cap (B \cup C) \subseteq (A \cap B) \cup C$.

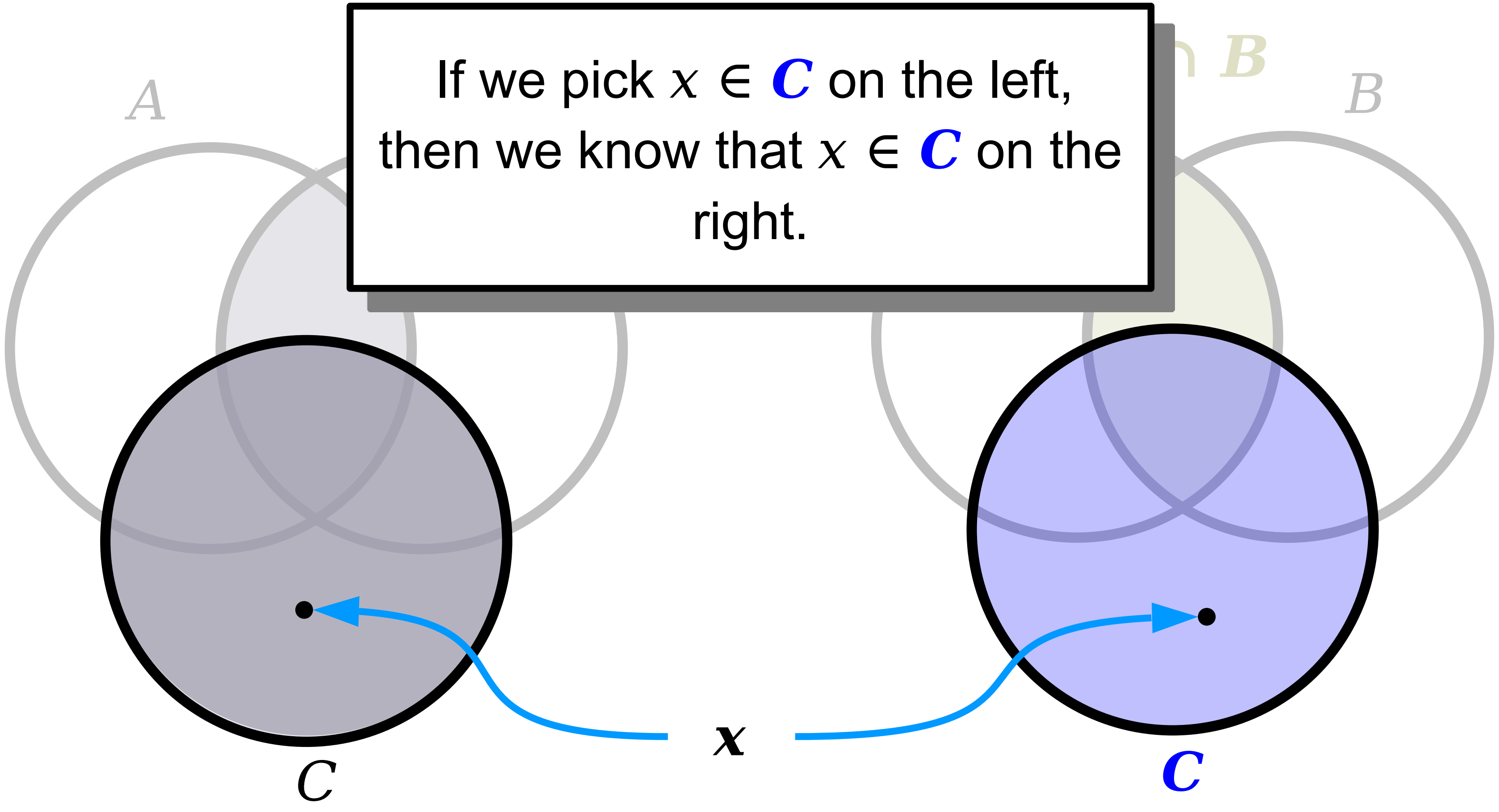
Let’s Draw Some Pictures!



Observation: Set ***C*** is in both of these shapes

Theorem: If A, B, and C are sets, then $(A \cup C) \cap (B \cup C) \subseteq (A \cap B) \cup C$.

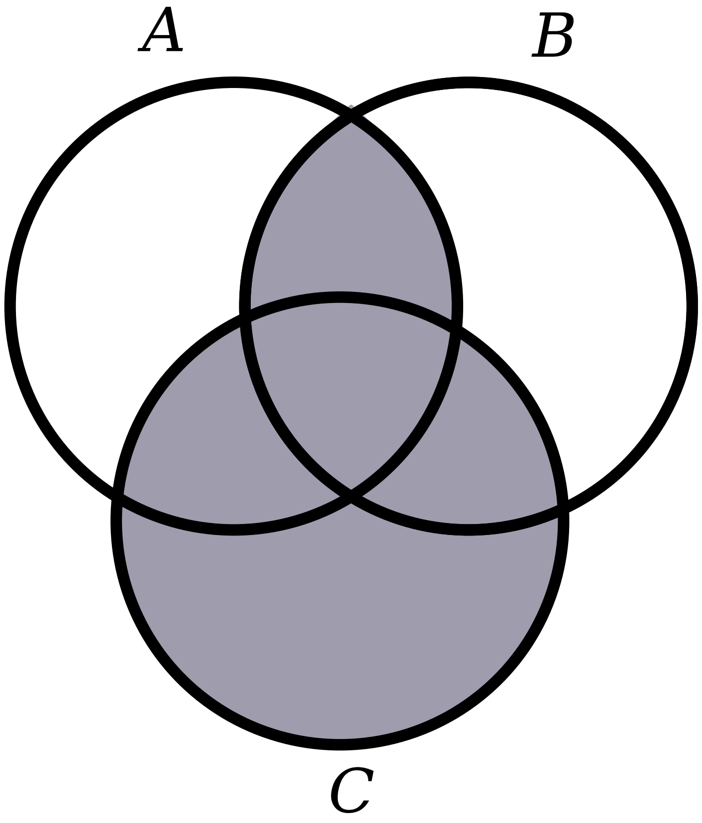
Let’s Draw Some Pictures!

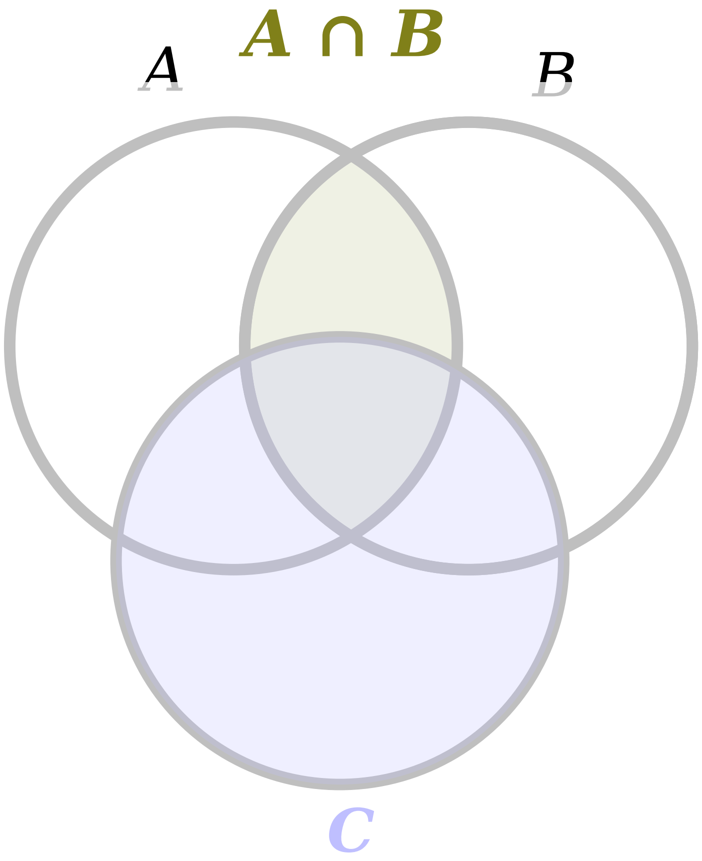


If we pick $x \in C$ on the left, then we know that $x \in C$ on the right.

Theorem: If A, B, and C are sets, then $(A \cup C) \cap (B \cup C) \subseteq (A \cap B) \cup C$.

Let’s Draw Some Pictures!





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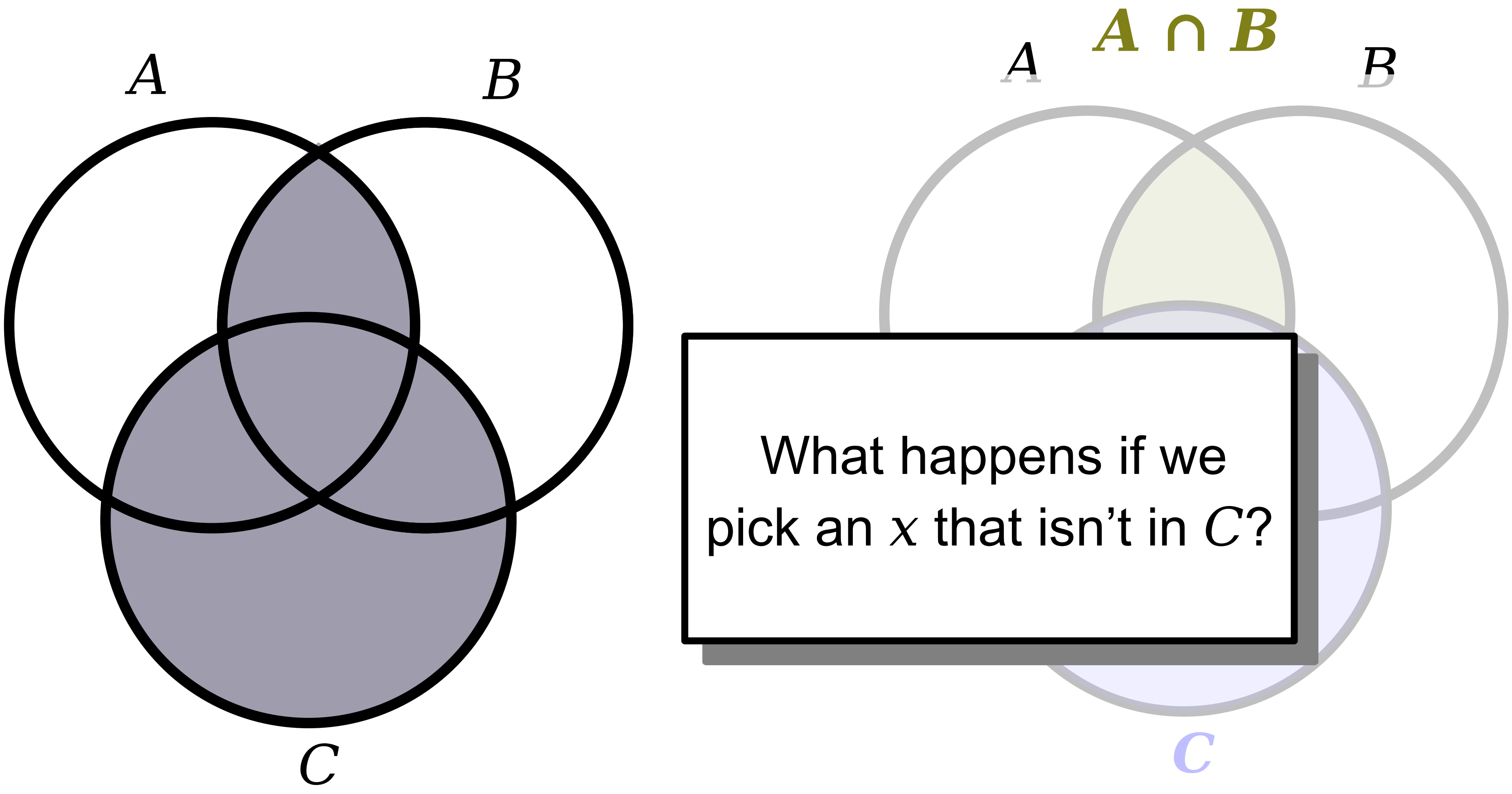
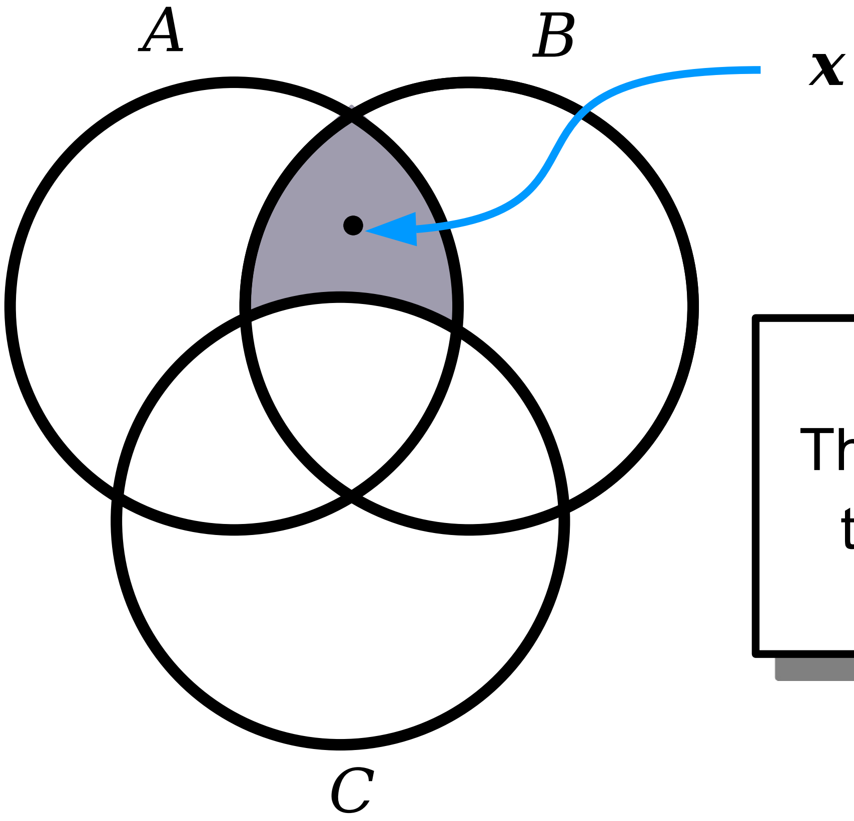
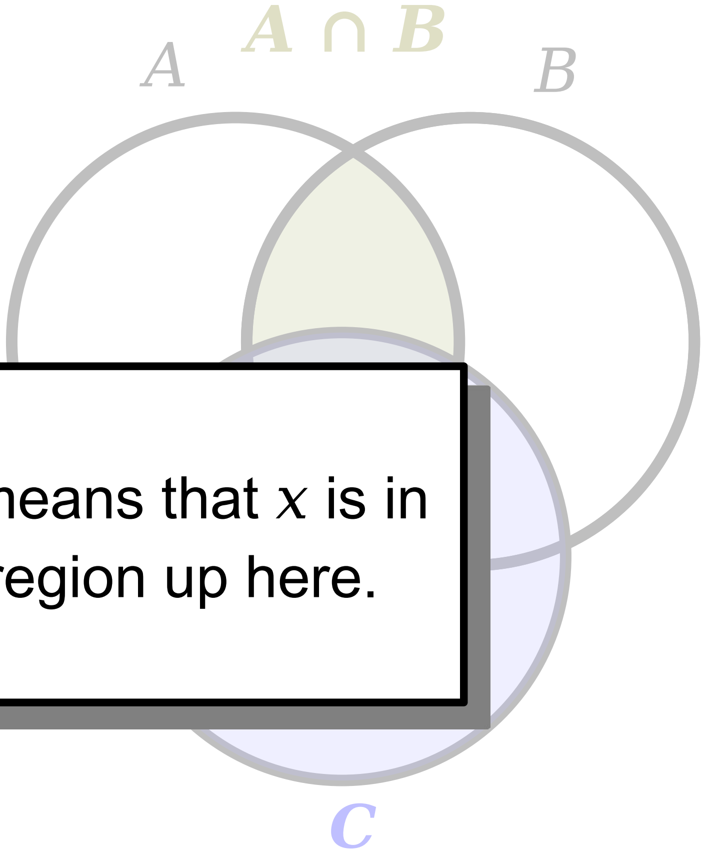


What happens if we pick an x that isn't in C?

Theorem: If A, B, and C are sets, then $(A \cup C) \cap (B \cup C) \subseteq (A \cap B) \cup C$.

Let's Draw Some Pictures!

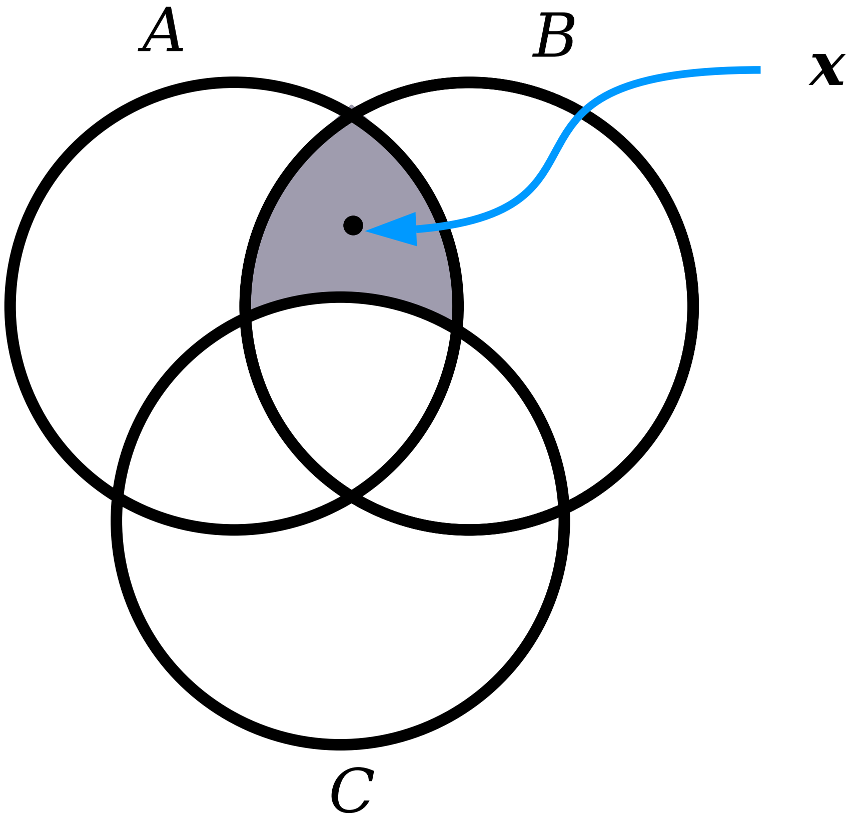


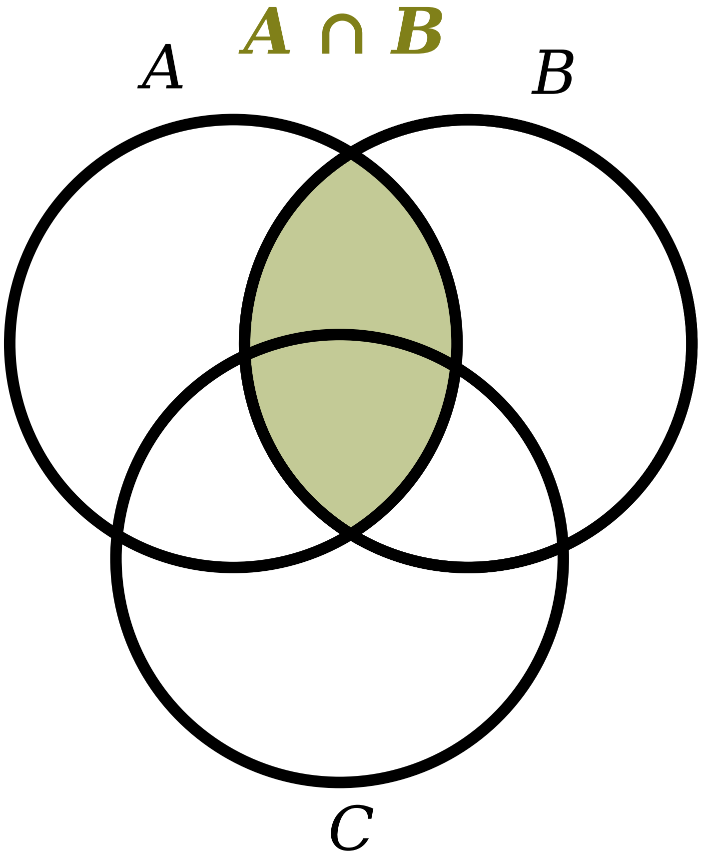


That means that x is in this region up here.

Theorem: If A, B, and C are sets, then $(A \cup C) \cap (B \cup C) \subseteq (A \cap B) \cup C$.

Let’s Draw Some Pictures!





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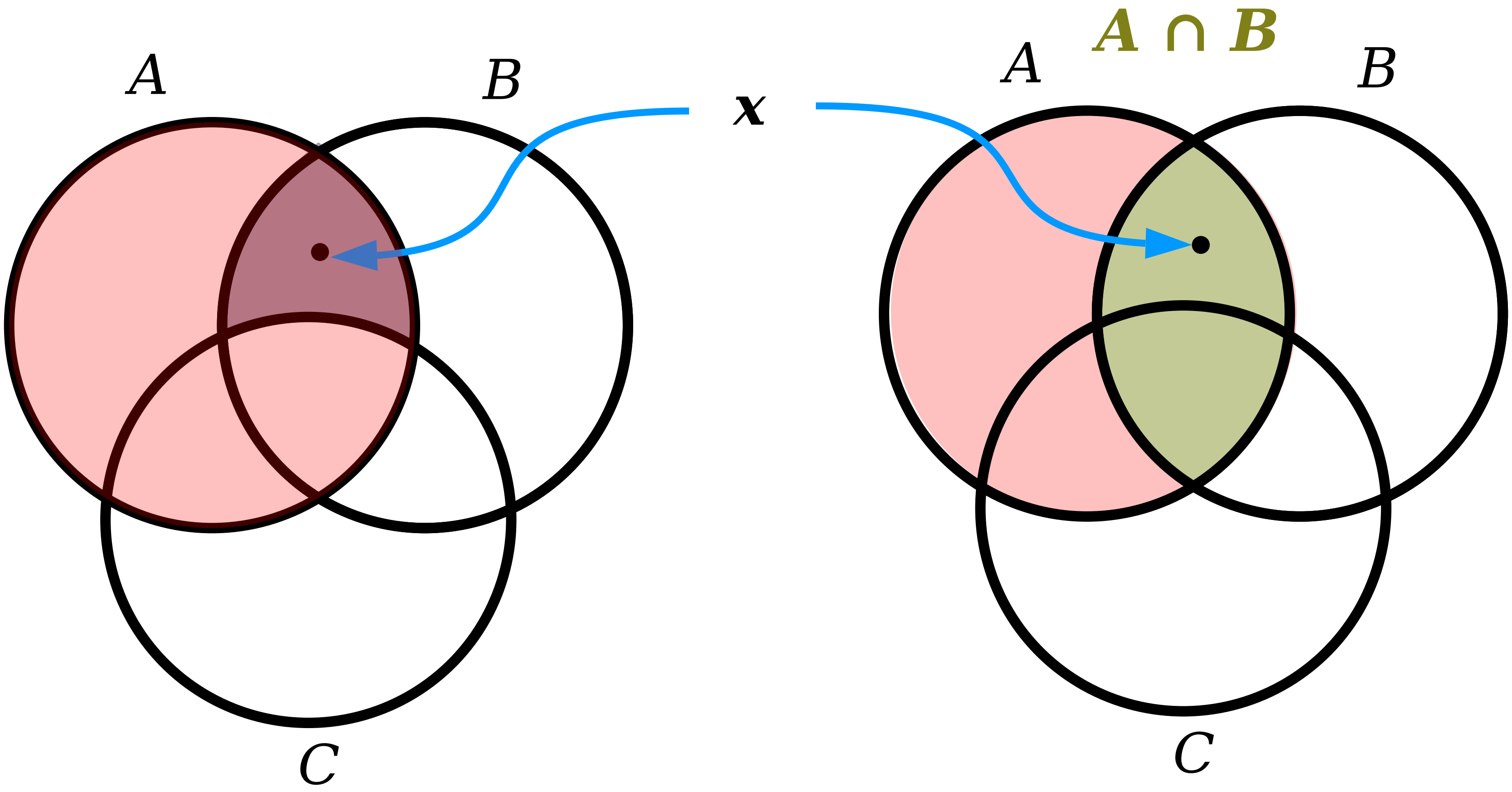

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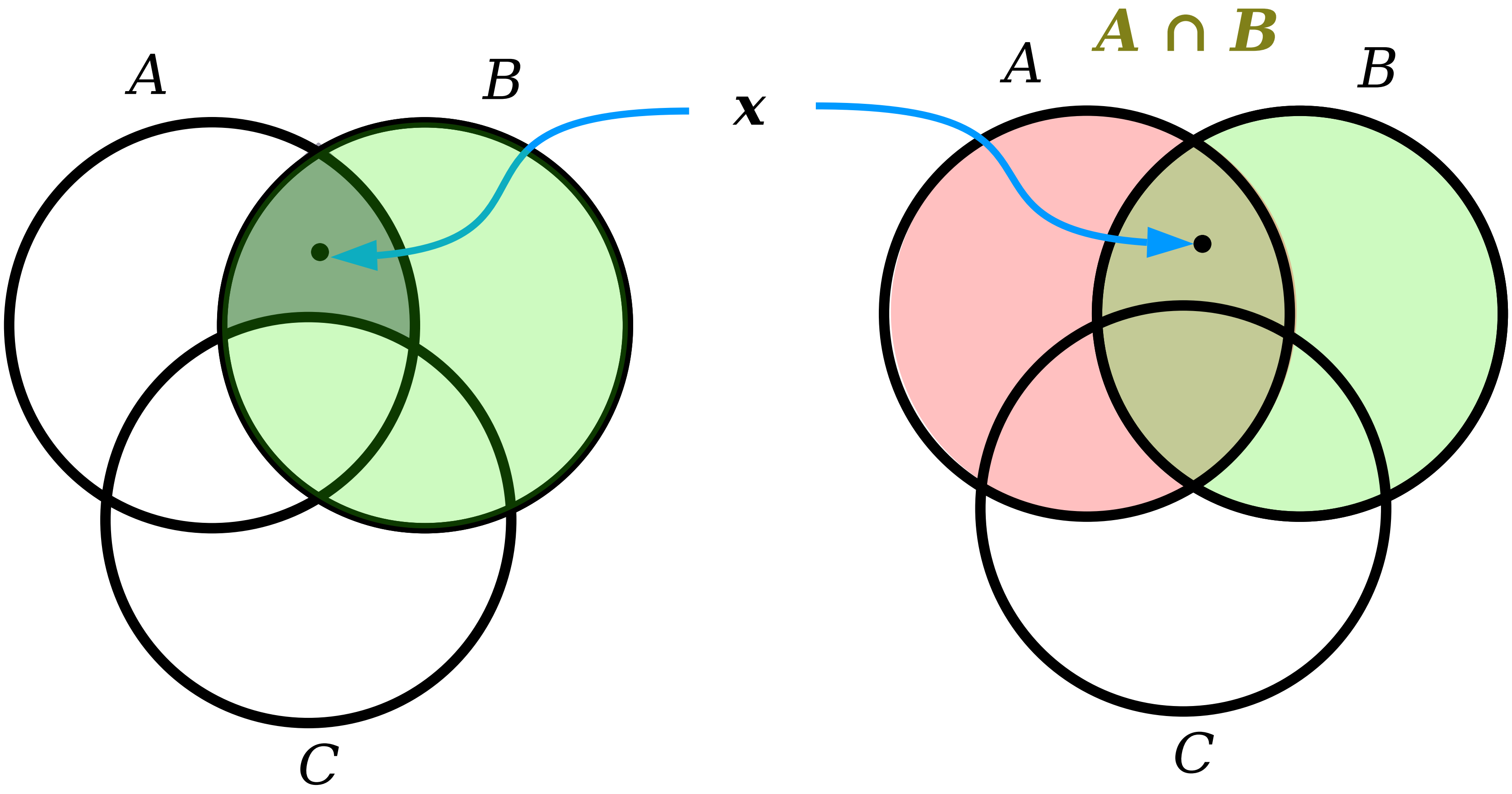

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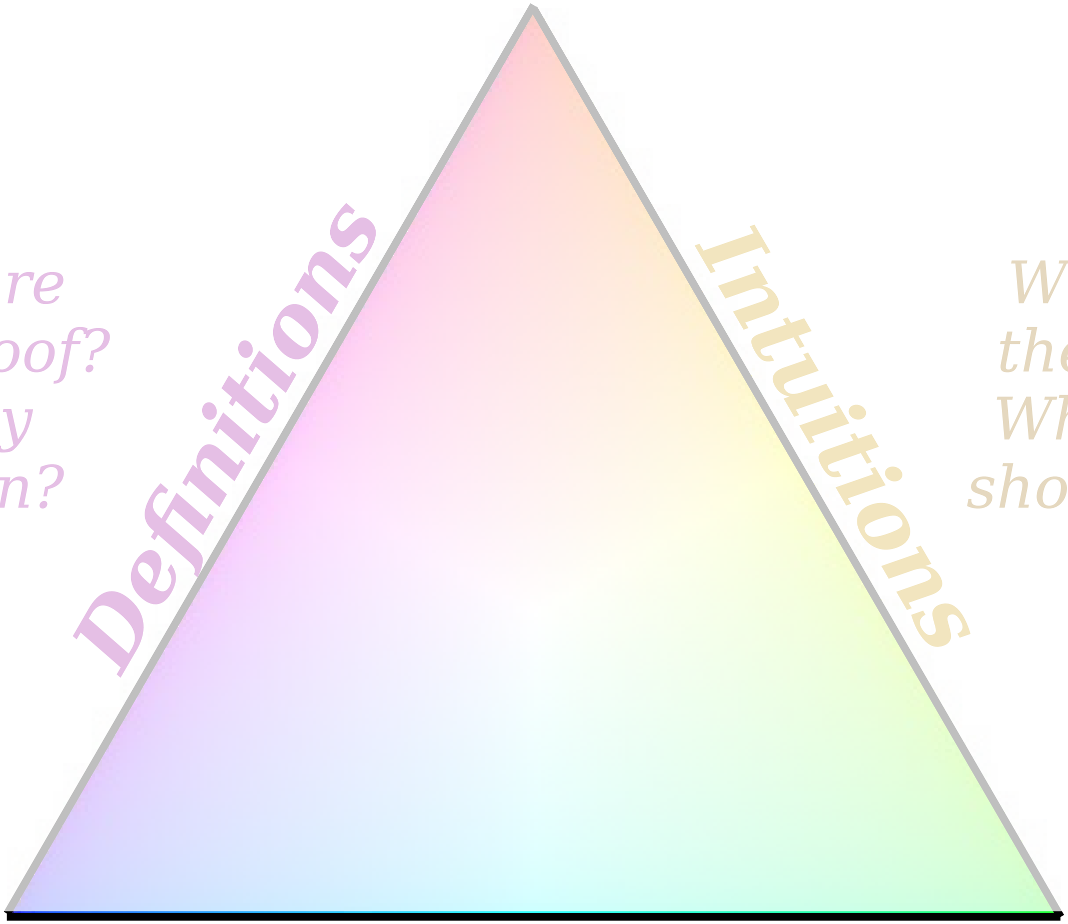

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Theorem: If A, B, and C are sets, then

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Proof:

Theorem: If A, B, and C are sets, then

$$(A \cup C) \cap (B \cup C) \subseteq (A \cap B) \cup C.$$

Proof: Pick any sets A, B, and C.

Theorem: If A, B, and C are sets, then $(A \cup C) \cap (B \cup C) \subseteq (A \cap B) \cup C$.

Proof: Pick any sets A, B, and C. Then, choose any element $x \in (A \cup C) \cap (B \cup C)$.

Theorem: If A, B, and C are sets, then $(A \cup C) \cap (B \cup C) \subseteq (A \cap B) \cup C$.

Proof: Pick any sets A, B, and C. Then, choose any element $x \in (A \cup C) \cap (B \cup C)$. We will prove that $x \in (A \cap B) \cup C$.

Theorem: If A, B, and C are sets, then $(A \cup C) \cap (B \cup C) \subseteq (A \cap B) \cup C$.

Proof: Pick any sets A, B, and C. Then, choose any element $x \in (A \cup C) \cap (B \cup C)$. We will prove that $x \in (A \cap B) \cup C$.

Since $x \in (A \cup C) \cap (B \cup C)$, we know that $x \in A \cup C$ and that $x \in B \cup C$.

Theorem: If A, B, and C are sets, then
$(A \cup C) \cap (B \cup C) \subseteq (A \cap B) \cup C$.

Proof: Pick any sets A, B, and C. Then, choose any element $x \in (A \cup C) \cap (B \cup C)$. We will prove that $x \in (A \cap B) \cup C$.

Since $x \in (A \cup C) \cap (B \cup C)$, we know that $x \in A \cup C$ and that $x \in B \cup C$. We now consider two cases.

***Case 1: $x \in C$*.**

***Case 2: $x \notin C$*.**

Theorem: If A, B, and C are sets, then $(A \cup C) \cap (B \cup C) \subseteq (A \cap B) \cup C$.

Proof: Pick any sets A, B, and C. Then, choose any element $x \in (A \cup C) \cap (B \cup C)$. We will prove that $x \in (A \cap B) \cup C$.

Since $x \in (A \cup C) \cap (B \cup C)$, we know that $x \in A \cup C$ and that $x \in B \cup C$. We now consider two cases.

Case 1: x ∈ C. This means $x \in (A \cap B) \cup C$ as well.

Case 2: x ∉ C.

Theorem: If A, B, and C are sets, then $(A \cup C) \cap (B \cup C) \subseteq (A \cap B) \cup C$.

Proof: Pick any sets A, B, and C. Then, choose any element $x \in (A \cup C) \cap (B \cup C)$. We will prove that $x \in (A \cap B) \cup C$.

Since $x \in (A \cup C) \cap (B \cup C)$, we know that $x \in A \cup C$ and that $x \in B \cup C$. We now consider two cases.

Case 1: $x \in C$. This means $x \in (A \cap B) \cup C$ as well.

Case 2: $x \notin C$. Because $x \in A \cup C$, we know that $x \in A$ or that $x \in C$.

Theorem: If A, B, and C are sets, then $(A \cup C) \cap (B \cup C) \subseteq (A \cap B) \cup C$.

Proof: Pick any sets A, B, and C. Then, choose any element $x \in (A \cup C) \cap (B \cup C)$. We will prove that $x \in (A \cap B) \cup C$.

Since $x \in (A \cup C) \cap (B \cup C)$, we know that $x \in A \cup C$ and that $x \in B \cup C$. We now consider two cases.

Case 1: $x \in C$. This means $x \in (A \cap B) \cup C$ as well.

Case 2: $x \notin C$. Because $x \in A \cup C$, we know that $x \in A$ or that $x \in C$. However, since we have $x \notin C$, we're left with $x \in A$.

Theorem: If A, B, and C are sets, then
$(A \cup C) \cap (B \cup C) \subseteq (A \cap B) \cup C$.

Proof: Pick any sets A, B, and C. Then, choose any element $x \in (A \cup C) \cap (B \cup C)$. We will prove that $x \in (A \cap B) \cup C$.

Since $x \in (A \cup C) \cap (B \cup C)$, we know that $x \in A \cup C$ and that $x \in B \cup C$. We now consider two cases.

Case 1: $x \in C$. This means $x \in (A \cap B) \cup C$ as well.

Case 2: $x \notin C$. Because $x \in A \cup C$, we know that $x \in A$ or that $x \in C$. However, since we have $x \notin C$, we're left with $x \in A$. By similar reasoning, from $x \in B \cup C$ we learn that $x \in B$.

Theorem: If A, B, and C are sets, then
$(A \cup C) \cap (B \cup C) \subseteq (A \cap B) \cup C$.

Proof: Pick any sets A, B, and C. Then, choose any element $x \in (A \cup C) \cap (B \cup C)$. We will prove that $x \in (A \cap B) \cup C$.

Since $x \in (A \cup C) \cap (B \cup C)$, we know that $x \in A \cup C$ and that $x \in B \cup C$. We now consider two cases.

Case 1: $x \in C$. This means $x \in (A \cap B) \cup C$ as well.

Case 2: $x \notin C$. Because $x \in A \cup C$, we know that $x \in A$ or that $x \in C$. However, since we have $x \notin C$, we're left with $x \in A$. By similar reasoning, from $x \in B \cup C$ we learn that $x \in B$.

Collectively, we've shown that $x \in A$ and that $x \in B$, so we see that $x \in A \cap B$. This means $x \in (A \cap B) \cup C$.

Theorem: If A, B, and C are sets, then
$(A \cup C) \cap (B \cup C) \subseteq (A \cap B) \cup C$.

Proof: Pick any sets A, B, and C. Then, choose any element $x \in (A \cup C) \cap (B \cup C)$. We will prove that $x \in (A \cap B) \cup C$.

Since $x \in (A \cup C) \cap (B \cup C)$, we know that $x \in A \cup C$ and that $x \in B \cup C$. We now consider two cases.

Case 1: $x \in C$. This means $x \in (A \cap B) \cup C$ as well.

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Collectively, we've shown that $x \in A$ and that $x \in B$, so we see that $x \in A \cap B$. This means $x \in (A \cap B) \cup C$.

In either case, we see that $x \in (A \cap B) \cup C$, which is what we needed to show.

Theorem: If A, B, and C are sets, then
$(A \cup C) \cap (B \cup C) \subseteq (A \cap B) \cup C$.

Proof: Pick any sets A, B, and C. Then, choose any element $x \in (A \cup C) \cap (B \cup C)$. We will prove that $x \in (A \cap B) \cup C$.

Since $x \in (A \cup C) \cap (B \cup C)$, we know that $x \in A \cup C$ and that $x \in B \cup C$. We now consider two cases.

Case 1: $x \in C$. This means $x \in (A \cap B) \cup C$ as well.

Case 2: $x \notin C$. Because $x \in A \cup C$, we know that $x \in A$ or that $x \in C$. However, since we have $x \notin C$, we're left with $x \in A$. By similar reasoning, from $x \in B \cup C$ we learn that $x \in B$.

Collectively, we've shown that $x \in A$ and that $x \in B$, so we see that $x \in A \cap B$. This means $x \in (A \cap B) \cup C$.

In either case, we see that $x \in (A \cap B) \cup C$, which is what we needed to show. ■

Theorem: If A, B, and C are sets, then $(A \cup C) \cap (B \cup C) \subseteq (A \cap B) \cup C$.

Proof: Pick any sets A, B, and C. Then, choose any element $x \in (A \cup C) \cap (B \cup C)$. We will prove that $x \in (A \cap B) \cup C$.

Since $x \in (A \cup C) \cap (B \cup C)$, we know that $x \in A \cup C$ and that $x \in B \cup C$. We now consider two cases.

Case ll.

Case $x \in A$

or e're

lef $B \cup C$

we

These are ***arbitrary choices***. Rather than specifying what ***A***, ***B***, and ***C*** are, we're signaling to the reader that they could, in principle, supply any choices of ***A***, ***B***, and ***C*** that they'd like.

Collectively, we've shown that $x \in A$ and that $x \in B$, so we see that $x \in A \cap B$. This means $x \in (A \cap B) \cup C$.

In either case, we see that $x \in (A \cap B) \cup C$, which is what we needed to show. ■

Theorem: If A, B, and C are sets, then
$(A \cup C) \cap (B \cup C) \subseteq (A \cap B) \cup C$.

Proof: Pick any sets A, B, and C. Then, choose any element $x \in (A \cup C) \cap (B \cup C)$. We will prove that $x \in (A \cap B) \cup C$.

Since x
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To prove that $S \subseteq T$:
Pick an arbitrary $x \in S$, then prove $x \in T$.

Notice that the statement of the theorem doesn't include any variable named *x*. We introduced this variable because that's what the definition says to do.

This is common in proofwriting. Always call back to the definition to make sure you're proving the right thing!

Theorem: If A, B, and C are sets, then $(A \cup C) \cap (B \cup C) \subseteq (A \cap B) \cup C$.

Proof: Pick any set
$x \in (A \cup C) \cap (B$.

Since $x \in (A \cup C$
that $x \in B \cup C$. W

As before, it's good to summarize what we established when splitting into cases.

Case 1: $x \in C$. This means $x \in (A \cap B) \cup C$ as well.

Case 2: $x \notin C$. Because $x \in A \cup C$, we know that $x \in A$ or that $x \in C$. However, since we have $x \notin C$, we're left with $x \in A$. By similar reasoning, from $x \in B \cup C$ we learn that $x \in B$.

Collectively, we've shown that $x \in A$ and that $x \in B$, so we see that $x \in A \cap B$. This means $x \in (A \cap B) \cup C$.

In either case, we see that $x \in (A \cap B) \cup C$, which is what we needed to show. ■

Theorem: If A, B, and C are sets, then
$(A \cup C) \cap (B \cup C) \subseteq (A \cap B) \cup C$.

Proof: Pick any sets A, B, and C. Then, choose any element $x \in (A \cup C) \cap (B \cup C)$. We will prove that $x \in (A \cap B) \cup C$.

Since $x \in (A \cup C) \cap (B \cup C)$, we know that $x \in A \cup C$ and that $x \in B \cup C$. We now consider two cases.

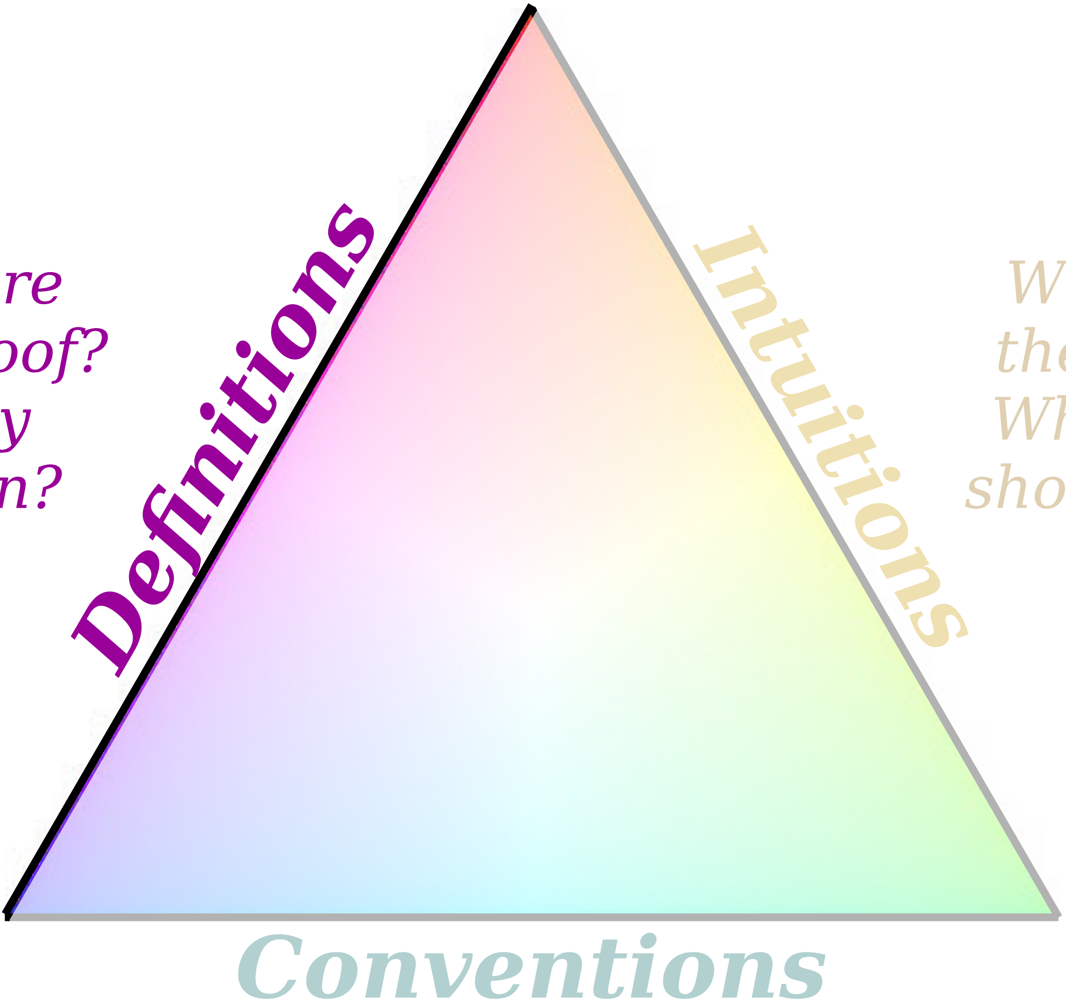
Case 1: $x \in C$. This means $x \in (A \cap B) \cup C$ as well.

Case 2: $x \notin C$. Because $x \in A \cup C$, we know that $x \in A$ or that $x \in C$. However, since we have $x \notin C$, we're left with $x \in A$. By similar reasoning, from $x \in B \cup C$ we learn that $x \in B$.

Collectively, we've shown that $x \in A$ and that $x \in B$, so we see that $x \in A \cap B$. This means $x \in (A \cap B) \cup C$.

In either case, we see that $x \in (A \cap B) \cup C$, which is what we needed to show. ■

Theorem: If A, B, and C are sets, then $(A \cup C) \cap (B \cup C) \ = \ (A \cap B) \cup C$.



What terms are
used in this proof?
What do they
formally mean?

What does this
theorem mean?
Why, intuitively,
should it be true?

What is the standard
format for writing a proof?
What are the techniques
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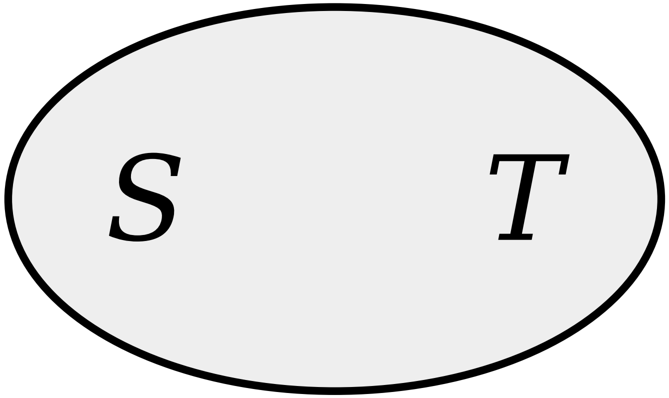
Set Equality



$S = T$

Definition: If S and T are sets, then $S = T$ if

$$S \subseteq T \quad \text{and} \quad T \subseteq S.$$

To prove that $S = T$:

Prove that $S \subseteq T$ and $T \subseteq S$.

If you know that $S = T$:

If you have an $x \in S$, you can conclude $x \in T$.
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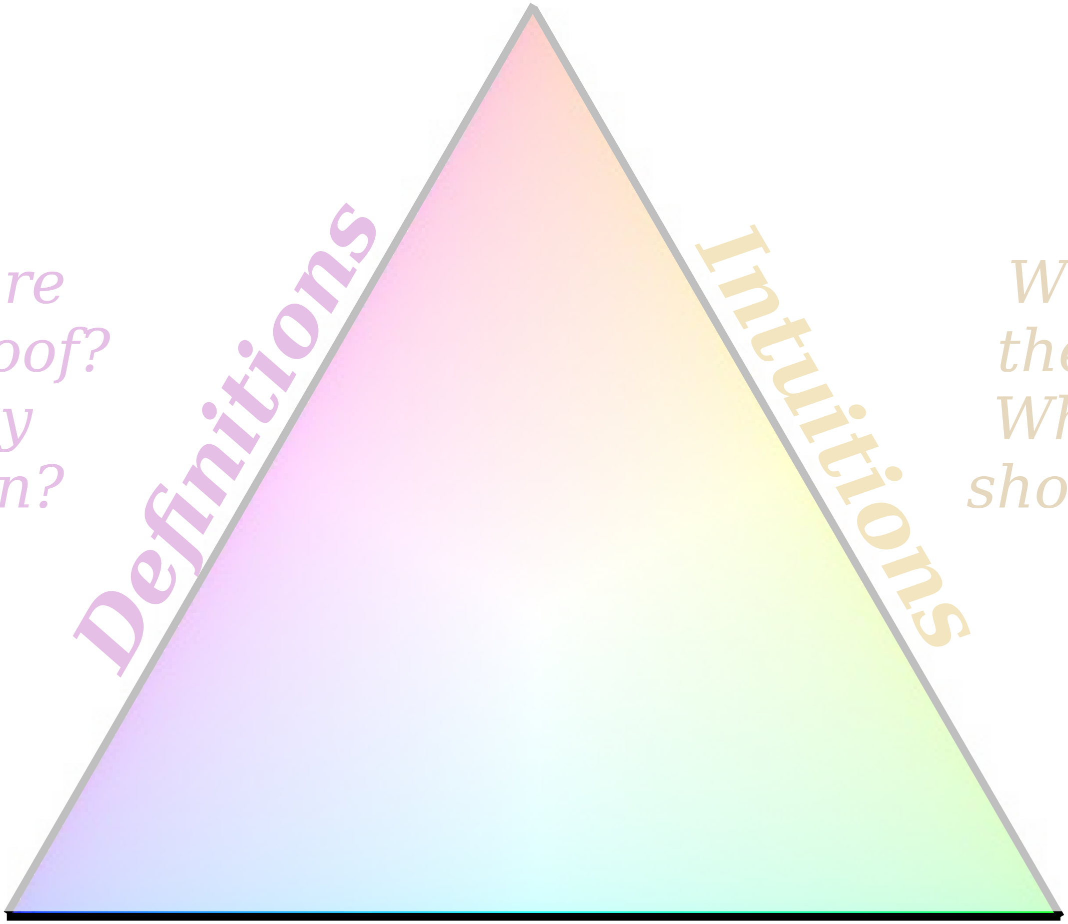
Theorem: If A, B, and C are sets, then $(A \cup C) \cap (B \cup C) = (A \cap B) \cup C$.



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Definitions

Intuitions

What does this
theorem mean?
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Conventions

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Since both (1) and (2) hold, we know that each of these two sets are subsets of one another, and therefore that the sets are equal.

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It is ***common*** for proofs in math to build on one another. That's how we make progress and make new discoveries!

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